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The impact of gamma usage processes on preventive maintenance policies under two-dimensional warranty

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ABSTRACT

This study considers a manufacturer performing preventive maintenance (PM) on a product according to a one- or two-dimensional (2-D) policy. The one-dimensional PM policy is based on either time or usage, while in the two-dimensional case, PM is scheduled based on both scales. The product carries a 2-D warranty that offers protection for a certain amount of time and usage. Its cumulative usage is continuously monitored by the manufacturer and is assumed to follow a gamma process. In this context, we first propose a doubly stochastic Poisson process model for product failures where the stochastic intensity is influenced by the gamma usage process in an additive manner. We then explicitly derive the expected total costs of the two one-dimensional PM policies using the concepts of first hitting times and gamma bridges. For the 2-D PM policy, we express the associated cost in terms of the value function of a dynamic programming model. In the numerical experiments, we show how the variability of the usage process affects the costs of the three PM policies and find that the optimal 2-D policy degenerates into a one-dimensional policy.

1. Introduction

Preventive maintenance (PM) refers to the routine maintenance of products that involves inspections, cleaning, lubrication, and parts replacement to prevent potential failures and unplanned downtime. There are two primary types of PM services in the aftermarket: traditional and flexible. In a traditional maintenance service, a manufacturer provides the same after-sales support to the entire customer population (e.g., the Volkswagen Jetta Maintenance Schedule). Unnecessary service costs or potential revenue losses may be incurred due to a lack of market segmentation. A flexible maintenance service has a menu of plan options for customers to choose from, as is the case with the Porsche Scheduled Maintenance Plan. Despite more flexibility in plan choice, each plan in the menu is designed based on the typical characteristics of the corresponding customer group, implying that a homogeneous service is still delivered in this segment. Additionally, some customers might not have a clear understanding of their own needs so that they are unable to identify their most suitable options.

In an attempt to address those shortcomings, many manufacturers such as truck firm Scania have embraced customized maintenance services with the help of Internet of Things (IoT) technology. Customization, as a business strategy, aims to sell products or services tailored to specific customer needs [1]. With permission, manufacturers can collect real-time sensor data during the use phase of a product to know how, where and by whom the product is being used. Such valuable information is then utilized to customize a maintenance service for each individual customer. Manufacturers benefit from this service by effectively engaging with customers and providing unique value to them. Finally, a closed-loop service system can be built to improve customer relations.

IoT data about product usage include not only basic variables like running time, the number of runs, and miles driven, but also variables regarding product conditions, operating environment, and customer behavior. This type of data has been traditionally difficult to obtain as most products, once sold, are beyond the control of manufacturers, and only limited field data is gathered during repair and maintenance processes [2]. In the IoT era, however, there are ample data on usage available at the individual level. Leveraging usage data to enhance decision-making is not easy. One reason is the random nature of usage data due to the variability in operations. A product may be used at various levels in any given period. For example, the daily mileage of a commercial vehicle fluctuates with customer demand. For products operating in the natural environment, such as wind turbines, their future usage is also generally uncertain. Consequently, the stochastic nature of usage data must be taken into account when making PM decisions.

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In this paper, we consider a manufacturer that performs PM on a product under a two-dimensional (2-D) warranty while continuously monitoring the cumulative usage of the product. We focus on exploring how its random cumulative usage affects the manufacturer's cost. We start by modeling the cumulative usage as a gamma process and then calculate the expected total costs of three PM policies with different triggering mechanisms. The first is a time-based PM policy where PM is performed at regular time intervals. It is the most commonly used PM policy in real-life situations. The second policy is usage-based and triggers a PM action once cumulative usage grows by a predetermined amount. According to Djamaludin et al. [3], this policy is specifically designed for products that tend to wear with use, such as automobile tires and aircraft engines. The last is a 2-D PM policy based on both time and usage. It requires that the next PM action occur after a certain amount of time or usage, whichever comes first. Considering the prevalence of 2-D PM in the automotive industry [4], we will study these three policies in the context of a 2-D warranty.

The rest of the paper is structured as follows. In Section 2, we review the relevant literature. In Section 3, we introduce gamma usage processes for modeling product failures. In Section 4, we calculate the expected total costs of the two one-dimensional PM policies. In Section 5, we construct a cost model of the 2-D PM policy using dynamic programming. In Section 6, we show by numerical experiments the effect of a stochastic usage process on the expected total costs of the three policies. We conclude in Section 7 with a summary and discussion.

2. Related literature

In the last decade, there has been considerable interest in planning PM activities under 2-D warranty. Wang and Xie [5] provided an excellent review of this literature. Wang et al. [6,7] both considered periodic PM within base and extended warranty periods. For used products sold with a 2-D warranty, Wang et al. [8] proposed a reliability improvement program involving either PM or upgrades, whereas Dai et al. [9] developed cost models for a used product that is upgraded prior to sale and then preventively maintained according to different policies. Several papers have incorporated nonconstant maintenance efforts in their PM models; that is, the effort or level of each PM action is not captured by a single variable [10–13].

The above-mentioned papers measured PM intervals in calendar time. Beyond that, PM can also be scheduled based on product usage, such as flight hours [14–16], but the existing warranty literature on usage-based PM remains limited. 2-D PM under a 2-D warranty was first analyzed by Wang and Su [4], who optimized the extent of PM as well as its time and usage intervals for the entire customer population. Following this framework, Su and Wang [17] examined the impact of the time of warranty purchase on a manufacturer's cost, while Wang et al. [18] considered the case where customers do not bring their products in for maintenance on time.

We note that product replacement is often seen as a means of PM (see, e.g., [19–21]). Using 2-D renewal theory, Yang and Nachlas [22] investigated a 2-D replacement policy when repair and replacement times are nonnegligible. Hu et al. [23] predicted the demand for spare parts under a 2-D replacement policy. There are also 2-D replacement models where the two scales are linearly combined, resulting in a policy that covers a triangular region in the 2-D plane (see, e.g., [24]). Frickenstein and Whitaker [25] extended the covered region to the class of lower sets.

Most models in the 2-D warranty literature assume that the cumulative usage of a product is a linear function of time with an unknown slope [26–28]. Although this assumption is supported by many statistical analyses of automobile warranty data, there are still reasons to investigate nonlinear usage processes. For the majority of drivers, what we observe is only an approximately linear sample path of a usage process, and a common usage rate distribution is not sufficient to customize PM services. For the rest, their usage paths tend to exhibit varying degrees of nonlinearity. As noted by Lawless et al. [2], it is short-term variations in usage rates that make usage paths not precisely linear. Several approaches appear in the literature to represent nonlinear usage paths, such as the accelerated failure time model [1,29], a weighted prediction model [30], and the logistic function [31]. Eliashberg et al. [31] treated cumulative usage as a stochastic function with respect to time. The functional form they chose reflects the decline in usage rate as a product ages.

We can also view product usage as a stochastic process. Singpurwalla and Wilson [32] described a usage process using three sets of nonnegative random variables, namely the lengths of time in busy and idle states as well as the usage rate at each busy period. De Jonge and Jakobsons [33] proposed a Markov switching model where busy and idle times are assumed to have exponential distributions. When periodically monitoring product usage, we can consider usage rates to be i.i.d. random variables by first dividing the planning horizon into equally spaced intervals, as in [13,19].

As for continuous monitoring, a gamma usage process is a very attractive alternative because of its mathematical tractability. Since it is a pure jump process, a countably infinite number of jumps occur in any finite time interval, suggesting that this process is appropriate for modeling the gradual accumulation of usage and can provide a good approximation of continuous usage paths [34]. Singpurwalla and Wilson [35] described the evolution of cumulative usage as Poisson, gamma, and Markov additive processes. Lawless and Crowder [34] fitted a joint model with random effects for recurrent events and stationary gamma usage processes in the context of automobile warranty claims. Their model checks indicated good agreement between the gamma usage process and actual warranty data. Pulcini [36] estimated a nonstationary gamma usage process model with heterogeneity using automobile failure data. For the application of gamma processes in the field of reliability, we refer interested readers to van Noortwijk [37], Singpurwalla [38], and Chen et al. [39].

Compared with the extant literature, the contribution of this research is threefold. First, to our knowledge, we are the first to study the three PM policies under a gamma usage process. The process variability not only leads to uncertainties in the number of PM actions, PM instants, and warranty end time, but also brings the possibility of performing PM in both dimensions under a 2-D PM policy. Second, to determine the expected total costs of the three policies, we develop a novel dynamic programming model with two state variables (time and cumulative usage) using the concepts of first hitting times, gamma bridges, and doubly stochastic Poisson processes. Third, 2-D PM in practical settings is typically delivered as a homogeneous service to the entire group of customers, whereas our focus is on a personalized PM service for a specific customer or usage process.

3. Model formulation

3.1. Gamma usage process

We assume that a usage process is determined by external factors, that is, the current product usage is unaffected by the previous history of failures [40]. This assumption is a reasonable approximation of reality because repair times are generally much shorter than the mean time between failures [2]. We view a usage process as a unidimensional concept captured by cumulative usage. Such a process is often approximated as linear when cumulative usage can only be obtained at the time of failure and is unavailable for products without failures. However, with the advent of the IoT, real-time tracking of usage has become feasible, enabling the recording of complete usage histories. This technological advancement allows analysts to use some stochastic processes in usage modeling. Both *between- and within-unit variations* in usage paths can be incorporated.

Table 1
Nomenclature.

Symbol	Definition
$M(t)$	Cumulative usage at time t
$\alpha t,\ \beta$	Shape function and rate parameter of the gamma process
$\lambda(t)$	Failure rate at time t
$\lambda_0(t)$	Baseline failure rate at time t
$\Lambda_0(t)$	$\int_0^t \lambda_0(s)\,\mathrm{d}s$
η	Usage deterioration efficient
$N(t)$	Number of failures up to time t
L	First failure time
$T,\ U$	Age and usage limits of the 2-D warranty
$\hat{T}$	Expiry time of the 2-D warranty
$\tau(x)$	First hitting time of the gamma process
ω_i	Instant of the ith PM action
ρ	Improvement factor of PM
c_r	Repair cost
c_p	PM cost
r	Usage rate
C_0	Expected repair cost when no PM action is taken
n	Number of PM actions under the time-based PM policy when $\hat{T}=T$
$\hat{n}$	n when no randomness is involved
$C_1(n)$	Expected total cost of the time-based PM policy
m	Number of PM actions under the usage-based PM policy when $M(T)>U$
$\hat{m}$	m when no randomness is involved
$C_2(m)$	Expected total cost of the usage-based PM policy
$J(x,y)$	Expected total cost from point (x,y) to the end of the 2-D warranty under the 2-D PM policy
$C_3(n,m)$	Expected total cost of the 2-D PM policy

We denote a product's cumulative usage at time t as $M(t)$ and its usage process as $\{M(t), t \geq 0\}$ (see Table 1 for a summary of notation). Our model assumes that the usage process $\{M(t), t \geq 0\}$ follows a stationary gamma process, which possesses the following properties:

- $M(0) = 0$;
- For any $0 \leq t_1 < t_2$, $M(t_2) - M(t_1) \sim \text{Gamma}(\alpha(t_2 - t_1), \beta)$;
- $\{M(t), t \geq 0\}$ has positive and independent increments.

Here the gamma distribution is parameterized in terms of a shape parameter $\alpha(t_2 - t_1) > 0$ and a rate parameter $\beta > 0$. In general, the shape parameter is the difference between the values of an increasing function passing through the origin at $t = t_2$ and $t = t_1$, but linearity is assumed for simplicity. The second property shows that the gamma process has stationary increments. The probability density function for $M(t_2) - M(t_1)$ can be expressed as:

$$f_{M(t_2)-M(t_1)}(x;\alpha(t_2-t_1),\beta) = \frac{\beta^{\alpha(t_2-t_1)}}{\Gamma(\alpha(t_2-t_1))} x^{\alpha(t_2-t_1)-1} e^{-\beta x} \tag{1}$$

for $x > 0$. Consider the increment of the gamma process over the interval $[0, t]$. Then we have $M(t) \sim \text{Gamma}(\alpha t, \beta)$. Its expectation $\mathbb{E}[M(t)] = \alpha t/\beta$ and variance $\text{Var}[M(t)] = \alpha t/\beta^2$ both increase linearly with time. Hence, the gamma process is able to model random fluctuations in a linear usage path. Multiplying $M(t)$ by a positive constant k, we have $kM(t) \sim \text{Gamma}(\alpha t, \beta/k)$.

Define $\tau(x) = \sup\{t : M(t) \leq x\}$ as the first hitting time of x by the gamma process. It is the time when the gamma usage process starting at zero first hits the cumulative usage x. The probability distribution function of this random variable is given by

$$F_{\tau(x)}(t) = P(\tau(x) \leq t) = P(M(t) \geq x) = \frac{\Gamma(\alpha t, \beta x)}{\Gamma(\alpha t)}, \tag{2}$$

where $\Gamma(a, y) = \int_y^\infty z^{a-1} e^{-z}\,\mathrm{d}z$ is the upper incomplete gamma function for $y \geq 0$ and $a > 0$. The probability density function of $\tau(x)$ is

$$f_{\tau(x)}(t) = \frac{\alpha}{\Gamma(\alpha t)} \int_{\beta x}^{\infty} \left(\ln z - \psi(\alpha t)\right) z^{\alpha t - 1} e^{-z}\,\mathrm{d}z, \tag{3}$$

where $\psi(y) = \mathrm{d}\ln\Gamma(y)/\mathrm{d}y$ is the Digamma function for $y > 0$.

Next we introduce the concept of a gamma bridge for later use. Similar to the definition of a Brownian bridge, a gamma bridge is a conditional stochastic process in the interval $[t_1, t_3]$, given the values of the process at both endpoints of the interval. For any $t_2 \in (t_1, t_3)$, we have two independent random variables: $M(t_2) - M(t_1) \sim \text{Gamma}(\alpha(t_2 - t_1), \beta)$ and $M(t_3) - M(t_2) \sim \text{Gamma}(\alpha(t_3 - t_2), \beta)$. From the properties of the gamma distribution, it follows that the first gamma random variable divided by the sum of the two gamma random variables has a beta distribution with parameters $\alpha(t_2 - t_1)$ and $\alpha(t_3 - t_2)$. That is, $\frac{M(t_2)-M(t_1)}{M(t_3)-M(t_1)} \sim \text{Beta}(\alpha(t_2-t_1), \alpha(t_3-t_2))$. Moreover, $\frac{M(t_2)-M(t_1)}{M(t_3)-M(t_1)}$ is independent of $M(t_3) - M(t_1)$. Since $M(t_2) - M(t_1)$ is equal to $M(t_3) - M(t_1)$ times a beta random variable, the conditional probability density of $M(t_2) - M(t_1)$ given $M(t_3) - M(t_1)$ is

$$\begin{aligned} & f_{M(t_2)-M(t_1)|M(t_3)-M(t_1)}(y \mid x) \\ &= \frac{\Gamma(\alpha(t_3-t_1))}{\Gamma(\alpha(t_2-t_1))\Gamma(\alpha(t_3-t_2))} \frac{1}{x} \left(\frac{y}{x}\right)^{\alpha(t_2-t_1)-1} \left(1 - \frac{y}{x}\right)^{\alpha(t_3-t_2)-1} \end{aligned} \tag{4}$$

for $0 < y < x$. In addition, we can obtain

$$\begin{aligned} \mathbb{E}\left[M(t_2) \mid M(t_1), M(t_3)\right] &= \mathbb{E}\left[M(t_2) \mid M(t_1), M(t_3) - M(t_1)\right] \\ &= M(t_1) + \mathbb{E}\left[\, M(t_2) - M(t_1) \mid M(t_1), M(t_3) - M(t_1) \,\right] \\ &= M(t_1) + \frac{t_2 - t_1}{t_3 - t_1}\left(M(t_3) - M(t_1)\right). \end{aligned} \tag{5}$$

The third equality holds because of the stationarity of the gamma process. We will frequently use Eq. (5) in the rest of the paper.

3.2. Doubly stochastic Poisson process

The product is subject to deterioration and wears with time and usage. We let $\lambda(t)$ denote its failure rate at time t. For analytical tractability, we assume that the effect of usage on reliability is captured by the following additive hazards model (see, e.g., [31,35]):

$$\lambda(t) = \lambda_0(t) + \eta M(t), \tag{6}$$

where $\lambda_0(t)$ is the baseline failure rate at time t and $\eta > 0$ is the usage deterioration factor. We use $\lambda_0(t)$ to describe the deterioration due to failure causes other than usage, such as rusting or damage by unexpected events. When environmental effects are small or the probability of adverse events tends to remain unchanged over time, one can adopt a constant baseline failure rate. The additive hazards model with a linear baseline failure rate has been widely used in the 2-D warranty literature, and He et al. [41] estimated it with truck warranty data. Aalen [42] discussed the merits of the additive hazards model over the usual proportional one (see, e.g., [43,44]).

We assume that a failure is corrected by minimal repair, that is, the failure rate is the same as it was just before the failure. Under this assumption, the failure process can be modeled by a Poisson process whose intensity is given by the above failure rate function. As the intensity is driven by a stochastic process, this Poisson process is itself a doubly stochastic Poisson process (also called a Cox process), which was initially proposed by Cox [45]. Note that once the realization of the intensity is given, it will become a nonhomogeneous Poisson process. Denote by $N(t)$ the number of failures up to time t. Then we have

$$\begin{aligned} \mathbb{E}[N(t)] &= \mathbb{E}\left[\int_0^t \lambda(s)\,\mathrm{d}s\right] = \int_0^t \mathbb{E}[\lambda(s)]\,\mathrm{d}s = \int_0^t \left(\lambda_0(s) + \frac{\eta\alpha s}{\beta}\right)\mathrm{d}s \\ &= \Lambda_0(t) + \frac{\eta\alpha t^2}{2\beta}, \end{aligned} \tag{7}$$

where $\Lambda_0(t) = \int_0^t \lambda_0(s)\,\mathrm{d}s$ is the baseline mean function. In the first equality, the integral of a gamma process with respect to time can be estimated by Riemann sums [35,46,47]. In the second equality, we interchange the order of expectation and integration. The third equality uses the fact that $M(s) \sim \text{Gamma}(\alpha s, \beta)$, which has mean $\alpha s/\beta$. The variance of the number of failures $N(t)$ is

$$\begin{aligned}\mathrm{Var}[N(t)] &= \mathbb{E}\left[\int_0^t \lambda(s)\,\mathrm{d}s\right] + \mathrm{Var}\left[\int_0^t \lambda(s)\,\mathrm{d}s\right]\\ &= \Lambda_0(t) + \frac{\eta\alpha t^2}{2\beta} + \int_0^t\int_0^t \mathbb{E}[\lambda(s)\lambda(z)]\,\mathrm{d}s\,\mathrm{d}z - \left(\Lambda_0(t) + \frac{\eta\alpha t^2}{2\beta}\right)^2\\ &= \Lambda_0(t) + \frac{\eta\alpha t^2}{2\beta} + \frac{\eta^2\alpha t^3}{3\beta^2}.\end{aligned} \tag{8}$$

The first equality follows from the law of total variance and the fact that $\mathrm{Var}[N(t) \mid \mathcal{F}_t] = \mathbb{E}[N(t) \mid \mathcal{F}_t] = \int_0^t \lambda(s)\,\mathrm{d}s$, where $\mathcal{F}_t = \sigma\{\lambda(s), s \le t\}$ is the sigma-algebra generated by the process $\{\lambda(s)\}_{s\le t}$. In the second equality, the last two terms represent the variance of the conditional expectation $\mathbb{E}[N(t) \mid \mathcal{F}_t]$, and $\mathbb{E}\left[(\mathbb{E}[N(t) \mid \mathcal{F}_t])^2\right] = \mathbb{E}\left[\int_0^t \lambda(s)\,\mathrm{d}s \int_0^t \lambda(z)\,\mathrm{d}z\right]$. The third equality holds because $\mathbb{E}[M(s)M(z)] = \alpha\min(s,z)/\beta^2 + \alpha^2 sz/\beta^2$. To see this, consider, for example, the case where $0 \le s \le z$. Then we can write $\mathbb{E}[M(s)M(z)]$ as $\mathbb{E}[M(s)(M(s) + M(z) - M(s))]$. That the doubly stochastic Poisson process is overdispersed follows from $\mathrm{Var}[N(t)] \ge \mathbb{E}[N(t)]$.

Another quantity of interest in the doubly stochastic Poisson process is the arrival time of the first failure, which we denote as L. When the manufacturer makes a replacement at failure, we need to specify the survival function $P(L > t)$ of this random variable for $t \ge 0$. From reliability theory, it follows that $P(L > t \mid \mathcal{F}_t) = \exp\left(-\int_0^t \lambda(s)\,\mathrm{d}s\right)$. By unconditioning, we obtain

$$\begin{aligned}P(L > t) &= \mathbb{E}\left[\exp\left(-\int_0^t \lambda(s)\,\mathrm{d}s\right)\right]\\ &= \exp\left(-\int_0^t (\lambda_0(s) + \alpha\ln(\beta+\eta s) - \alpha\ln\beta)\,\mathrm{d}s\right)\\ &= \exp\big(\alpha t - \Lambda_0(t)\big)\left(\frac{\beta}{\eta t+\beta}\right)^{\alpha(t+\beta/\eta)}.\end{aligned} \tag{9}$$

The second equality uses the following result established in [48]:

$$\begin{aligned}\mathbb{E}\left[\exp\left(-\int_0^t M(s)\,\mathrm{d}s\right)\right] &= \exp\left(-\int_0^t (\alpha\ln(\beta+s) - \alpha\ln\beta)\,\mathrm{d}s\right)\\ &= \exp(\alpha t)\left(\frac{\beta}{t+\beta}\right)^{\alpha(t+\beta)}.\end{aligned} \tag{10}$$

We refer the reader to Kebir [48] for the proof of this result. By Eq. (9), the product behaves, in an average sense, as if it had an increasing deterministic failure rate function given by $\lambda_0(s) + \alpha\ln(\beta+\eta s) - \alpha\ln\beta$.

3.3. PM effect

Before determining the cost of a PM policy, we need to model the effect of PM. There are two commonly used modeling approaches in the literature: reducing the failure rate of a product indirectly or directly [49]. The first approach uses the notion of virtual age and applies to the case of deterministic failure rates. By integrating a failure rate function over intervals of virtual age, we can find the expected number of failures. However, when dealing with stochastic failure rates due to gamma usage processes, for a usage-based PM policy introduced in Section 4.2, it is challenging to specify which PM interval a product's virtual age after PM falls into. Therefore, we adopt the second modeling approach, and specifically the ARI_1 model in [49], to ensure analytical tractability.

We assume that PM can reduce the increment in the failure rate since the most recent PM action (rather than the current failure rate level) by a constant factor. Let ω_i be the instant at which the ith PM action is performed, with $\omega_0 = 0$. Then we have $\lambda(\omega_{i+1}) = \lambda(\omega_i) + (1-\rho)(\lambda(\omega_{i+1}^-) - \lambda(\omega_i))$, where $0 \le \rho \le 1$ denotes the improvement factor and $\lambda(\omega_{i+1}^-) = \lim_{t\to\omega_{i+1}^-}\lambda(t)$. Solving the recursion yields

$$\begin{aligned}\lambda(\omega_i) &= \lambda(0) + (1-\rho)\big(\lambda_0(\omega_i) + \eta M(\omega_i) - \lambda_0(0)\big)\\ &= \rho\lambda_0(0) + (1-\rho)\big(\lambda_0(\omega_i) + \eta M(\omega_i)\big).\end{aligned} \tag{11}$$

Even though $\lambda(\omega_i) \ne \lambda_0(\omega_i) + \eta M(\omega_i)$ when PM is present, Eq. (6) can still be used to derive the failure rate increment $\lambda(\omega_{i+1}^-) - \lambda(\omega_i)$. For any $\omega_i \le t < \omega_{i+1}$, we have

$$\lambda(t) = \lambda_0(t) + \eta M(t) - \rho\big(\lambda_0(\omega_i) + \eta M(\omega_i) - \lambda_0(0)\big). \tag{12}$$

This equation indicates that we just need to subtract a portion of the total increase in the failure rate from the beginning to the last PM instant.

4. One-dimensional PM policies

In this section, we examine two one-dimensional PM policies, one based on time and the other based on usage. PM is conducted under a 2-D warranty with an age limit T and a usage limit U. The costs of the two policies are closely related to the end of the 2-D warranty. We let this point be $\widehat{T} = \min\{\tau(U), T\}$, where $\tau(U)$ corresponds to the time when the product's usage process first hits U. Its probability distribution function is

$$F_{\widehat{T}}(t) = P\big(\widehat{T} \le t\big) = \begin{cases} P\big(\tau(U) \le t\big), & \text{if } t < T,\\ 1, & \text{if } t \ge T.\end{cases} \tag{13}$$

The manufacturer incurs a fixed cost of c_r for each repair operation. If we let C_0 denote the expected repair cost of the manufacturer when there is no PM, then we have $C_0 = c_r\mathbb{E}[N(\widehat{T})]$, where

$$\begin{aligned}\mathbb{E}\left[N\big(\widehat{T}\big)\right] &= \int_0^\infty \mathbb{E}\left[N\big(\widehat{T}\big) \mid \tau(U) = t\right] f_{\tau(U)}(t)\,\mathrm{d}t\\ &= \int_0^T \mathbb{E}[N(t) \mid \tau(U) = t]\, f_{\tau(U)}(t)\,\mathrm{d}t\\ &\quad + \int_T^\infty \mathbb{E}[N(T) \mid \tau(U) = t]\, f_{\tau(U)}(t)\,\mathrm{d}t\\ &= \int_0^T \mathbb{E}\left[\int_0^t \big(\lambda_0(s) + \eta M(s)\big)\,\mathrm{d}s \;\middle|\; M(t) = U\right] f_{\tau(U)}(t)\,\mathrm{d}t\\ &\quad + \int_T^\infty \mathbb{E}\left[\int_0^T \big(\lambda_0(s) + \eta M(s)\big)\,\mathrm{d}s \;\middle|\; M(t) = U\right] f_{\tau(U)}(t)\,\mathrm{d}t\\ &= \int_0^T \int_0^t \mathbb{E}[\lambda_0(s) + \eta M(s) \mid M(t) = U]\,\mathrm{d}s\, f_{\tau(U)}(t)\,\mathrm{d}t\\ &\quad + \int_T^\infty \int_0^T \mathbb{E}[\lambda_0(s) + \eta M(s) \mid M(t) = U]\,\mathrm{d}s\, f_{\tau(U)}(t)\,\mathrm{d}t\\ &= \int_0^T \left(\Lambda_0(t) + \frac{\eta U t}{2}\right) f_{\tau(U)}(t)\,\mathrm{d}t\\ &\quad + \int_T^\infty \left(\Lambda_0(T) + \frac{\eta U T^2}{2t}\right) f_{\tau(U)}(t)\,\mathrm{d}t.\end{aligned} \tag{14}$$

In the second equality, we use the law of total expectation. The first term on the right-hand side corresponds to the case where the usage limit is reached before time T. In this case, the 2-D warranty ends at time $\tau(U)$, and the number of failures occurring before time $\tau(U)$ is counted. The second term represents the case where the 2-D warranty expires at time T. The fourth equality follows by interchanging the order of conditional expectation and integration. The fifth equality is obtained from Eq. (5). Note that the probability density function for $\tau(U)$ is obtained from Eq. (3) by letting $x = U$. The expected number of failures occurring by a given time and usage can be derived in a way similar to $\mathbb{E}[N\big(\widehat{T}\big)]$.

4.1. Time-based PM policy

The time-based PM policy we consider is periodic, and PM takes place every $T/(n+1)$ units of time. Under this policy, the ith PM instant ω_i equals $iT/(n+1)$ for $1 \le i \le \lceil (n+1)\widehat{T}/T \rceil - 1$. As an illustration, Fig. 1 plots two gamma usage processes when $n = 2$. Since $\widehat{T} = T$

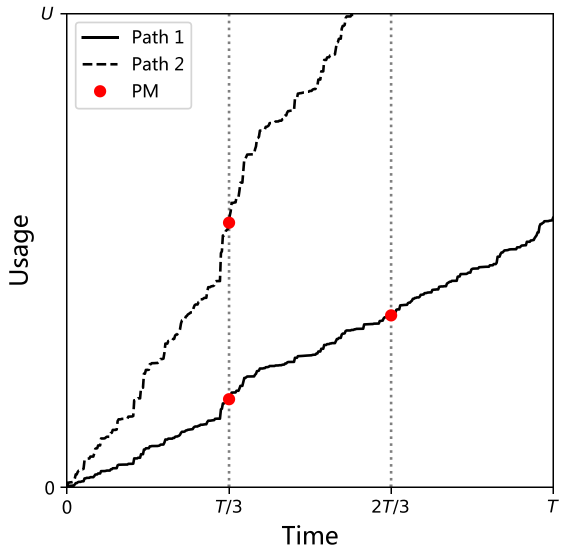

Fig. 1. Illustration of the time-based PM policy ($n = 2$).

in the case of path 1, the number of PM actions is n. (Note that the manufacturer does not perform PM at the end of the 2-D warranty.) For path 2, this number is less than n. Although simulated from pure jump processes, both usage paths still appear to be smooth. However, there also exist occasional noticeable jumps due to heavy use, which increase the uncertainties in the cumulative usage and failure rate at each PM instant.

Since some auto brands (such as BMW, Volvo, and Jaguar) provide a comprehensive free maintenance program to their customers, we assume that the manufacturer bears all the PM costs in the warranty period. The cost of each PM task is denoted as c_p and is assumed to be independent of product states. Both the number of PM actions and the total PM cost are random because the cumulative usage follows a gamma process. Let $C_1(n)$ be the expected total cost of the time-based PM policy. This cost consists of the expected repair and PM costs. The decision variable $n \geq 0$ is an integer. The cost function $C_1(n)$ can be written as

$$\begin{aligned} C_1(n) = c_r \mathbb{E}\left[\sum_{j=0}^{\lceil (n+1)\hat{T}/T \rceil - 2} \int_{\frac{jT}{n+1}}^{\frac{(j+1)T}{n+1}} \lambda(s)\,\mathrm{d}s + \int_{\frac{(\lceil (n+1)\hat{T}/T \rceil - 1)T}{n+1}}^{\hat{T}} \lambda(s)\,\mathrm{d}s \right] \\ + c_p \sum_{i=0}^{n} \left(F_{\tau(U)}\left(\frac{(i+1)T}{n+1}\right) - F_{\tau(U)}\left(\frac{iT}{n+1}\right) \right) i \\ + c_p \left(1 - F_{\tau(U)}(T)\right) n. \end{aligned} \tag{15}$$

The first term in the above equation represents the expected repair cost and is described in further detail in Appendix. The second and third terms correspond to the expected PM cost, and we need to pay attention to the time when the usage path first hits the usage limit of the 2-D warranty. The realized value of $\hat{T}$ determines the number of PM operations performed.

If there is no randomness in usage, a stationary gamma usage process will be reduced to a linear usage process, with its path being a straight line. The slope of this line is referred to as the usage rate and is denoted by r. The constant usage rate r is known because this information can be collected by sensors. Given r, we can express the cumulative usage at time t as $M(t) = rt$. The expected total cost of the time-based PM policy depends on the usage rate r. When $0 < r \leq U/T$, we have

$$\begin{aligned} C_1(n) &= c_p n + c_r \sum_{j=0}^{n} \int_{\frac{jT}{n+1}}^{\frac{(j+1)T}{n+1}} \lambda(s)\,\mathrm{d}s \\ &= c_p n + c_r \sum_{j=0}^{n} \left(\Lambda_0\left(\frac{(j+1)T}{n+1}\right) - \Lambda_0\left(\frac{jT}{n+1}\right) + \frac{(2j+1)\eta r T^2}{2(n+1)^2} \right. \\ &\quad \left. - \frac{\rho T}{n+1}\left(\lambda_0\left(\frac{jT}{n+1}\right) + \frac{j\eta r T}{n+1} - \lambda_0(0) \right) \right). \end{aligned} \tag{16}$$

The first equality follows from Eq. (15) by noting that $\hat{T} = T$ and $\tau(U) = U/r \geq T$. Similarly, when $r > U/T$, we have $\hat{T} = \tau(U) = U/r < T$. In this case, the cost function $C_1(n)$ is given by

$$\begin{aligned} C_1(n) &= c_p \hat{n} + c_r \sum_{j=0}^{\hat{n}-1} \int_{\frac{jT}{n+1}}^{\frac{(j+1)T}{n+1}} \lambda(s)\,\mathrm{d}s + c_r \int_{\frac{\hat{n}T}{n+1}}^{\frac{U}{r}} \lambda(s)\,\mathrm{d}s \\ &= c_p \hat{n} + c_r \sum_{j=0}^{\hat{n}-1} \left(\Lambda_0\left(\frac{(j+1)T}{n+1}\right) - \Lambda_0\left(\frac{jT}{n+1}\right) + \frac{(2j+1)\eta r T^2}{2(n+1)^2} \right. \\ &\quad \left. - \frac{\rho T}{n+1}\left(\lambda_0\left(\frac{jT}{n+1}\right) + \frac{j\eta r T}{n+1} - \lambda_0(0) \right) \right) \\ &\quad + c_r \left(\Lambda_0\left(\frac{U}{r}\right) - \Lambda_0\left(\frac{\hat{n}T}{n+1}\right) \right. \\ &\quad + \frac{\eta r}{2}\left(\frac{U^2}{r^2} - \frac{\hat{n}^2 T^2}{(n+1)^2} \right) - \rho\left(\lambda_0\left(\frac{\hat{n}T}{n+1}\right) \right. \\ &\quad \left. \left. + \eta r\left(\frac{\hat{n}T}{n+1}\right) - \lambda_0(0) \right)\left(\frac{U}{r} - \frac{\hat{n}T}{n+1} \right) \right), \end{aligned} \tag{17}$$

where $\hat{n} = \left\lceil \frac{(n+1)U}{rT} \right\rceil - 1$ designates the number of PM actions and $\hat{n}T/(n+1) < \tau(U) \leq (\hat{n}+1)T/(n+1)$.

4.2. Usage-based PM policy

In addition to the time dimension, we can also characterize a PM policy using the usage dimension. The usage-based PM policy considered divides the usage interval $[0, U]$ into $m+1$ equal parts, making the amount of usage between two consecutive PM actions $U/(m+1)$. The corresponding time interval is of length $\tau\left(\frac{U}{m+1}\right)$, which is the time when a gamma process starting from zero first hits $U/(m+1)$. Define $\left\{\tau_i\left(\frac{U}{m+1}\right)\right\}_{i=1}^{m+1}$ as an independent sequence of such first hitting times. Then the ith PM instant ω_i can be expressed as $\omega_i = \tau\left(\frac{iU}{m+1}\right) = \sum_{j=1}^{i} \tau_j\left(\frac{U}{m+1}\right)$ for $1 \leq i \leq \lceil (m+1)M(\hat{T})/U \rceil - 1$.

We denote by $C_2(m)$ the expected total cost of the usage-based PM policy. The decision variable $m \geq 0$ is an integer. The expression for $C_2(m)$ can be written as:

$$\begin{aligned} C_2(m) = \mathbb{E}\left[\mathbb{E}\left[\text{repair cost} \,\middle|\, \tau_1\left(\frac{U}{m+1}\right), \ldots, \tau_{m+1}\left(\frac{U}{m+1}\right) \right] \right] \\ + c_p \sum_{i=0}^{m} \left(F_{M(T)}\left(\frac{(i+1)U}{m+1}\right) - F_{M(T)}\left(\frac{iU}{m+1}\right) \right) i \\ + c_p \left(1 - F_{M(T)}(U)\right) m, \end{aligned} \tag{18}$$

where $F_{M(T)}$ is the cumulative density function of the gamma-distributed variable $M(T)$. It is necessary to condition on the sequence of first hitting times when calculating the expected repair cost. This expression can be further expanded, as shown in Appendix. The key idea in computing the expected PM cost is to determine how many usage intervals the gamma process goes through during the 2-D warranty—that is, how many times the manufacturer has performed PM. For example, we can see from Fig. 2 that the total number of PM actions is m when $M(T) > U$. The figure also reveals that the usage-based policy leads to nonperiodic PM schedules. Compared with the time-based policy, this policy leaves the manufacturer facing uncertainty about PM instants rather than the cumulative usage at each instant.

We next derive the expected total cost of the usage-based PM policy if there is no randomness involved in the usage process. When $0 < r \leq U/T$, we have

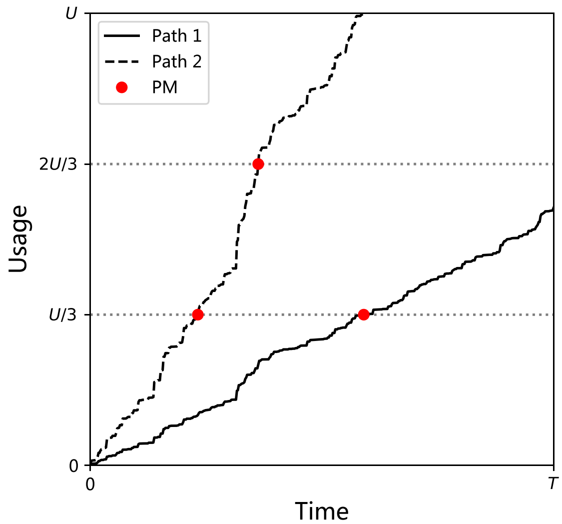

Fig. 2. Illustration of the usage-based PM policy ($m = 2$).

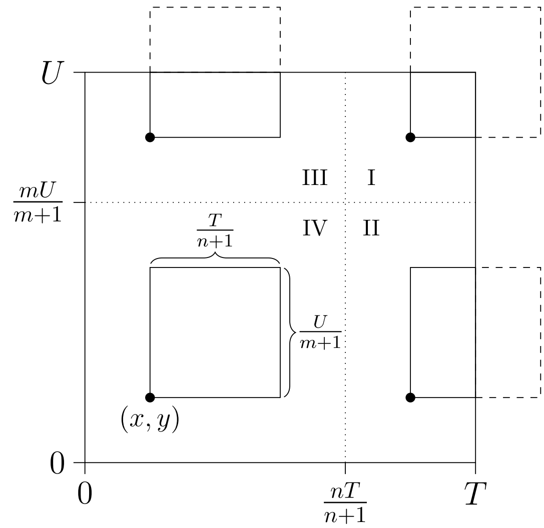

Fig. 3. Four cases of the recursive equation for $J(x, y)$.

$$
\begin{aligned}
C_2(m) &= c_p \hat{m} + c_r \sum_{j=0}^{\hat{m}-1} \int_{\frac{jU}{(m+1)r}}^{\frac{(j+1)U}{(m+1)r}} \lambda(s)\,\mathrm{d}s + c_r \int_{\frac{\hat{m}U}{(m+1)r}}^{T} \lambda(s)\,\mathrm{d}s \\
&= c_p \hat{m} + c_r \sum_{j=0}^{\hat{m}-1} \left(\Lambda_0\left(\frac{(j+1)U}{(m+1)r}\right) - \Lambda_0\left(\frac{jU}{(m+1)r}\right) + \frac{(2j+1)\eta r U^2}{2(m+1)^2 r^2} \right. \\
&\quad \left. - \frac{\rho U}{(m+1)r}\left(\lambda_0\left(\frac{jU}{(m+1)r}\right) + \frac{j\eta r U}{(m+1)r} - \lambda_0(0)\right)\right) \\
&\quad + c_r \left(\Lambda_0(T) - \Lambda_0\left(\frac{\hat{m}U}{(m+1)r}\right) \right. \\
&\quad + \frac{\eta r}{2}\left(T^2 - \frac{\hat{m}^2 U^2}{(m+1)^2 r^2}\right) - \rho\left(\lambda_0\left(\frac{\hat{m}U}{(m+1)r}\right) \right. \\
&\quad \left. + \eta r \left(\frac{\hat{m}U}{(m+1)r}\right) - \lambda_0(0)\right) \\
&\quad \left. \cdot \left(T - \frac{\hat{m}U}{(m+1)r}\right)\right),
\end{aligned}
\tag{19}
$$

where $\hat{m} = \left\lceil \frac{(m+1)rT}{U} \right\rceil - 1$ is the number of PM actions and $\hat{m}U/(m+1) < rT \le (\hat{m}+1)U/(m+1)$. The first equality can be obtained from Eqs. (18) and (A.2) by noting that $M(t) = rT \le U$ and $\sum_{j=1}^{\hat{m}} \tau_j \left(\frac{U}{m+1}\right) < T \le \sum_{j=1}^{\hat{m}+1} \tau_j \left(\frac{U}{m+1}\right)$. Similarly, when $r > U/T$, we have $M(t) = rT > U$ and $\sum_{j=1}^{m+1} \tau_j \left(\frac{U}{m+1}\right) = U/r < T$. In this case, the cost function $C_2(m)$ is given by

$$
\begin{aligned}
C_2(m) &= c_p m + c_r \sum_{j=0}^{m} \int_{\frac{jU}{(m+1)r}}^{\frac{(j+1)U}{(m+1)r}} \lambda(s)\,\mathrm{d}s \\
&= c_p m + c_r \sum_{j=0}^{m} \left(\Lambda_0\left(\frac{(j+1)U}{(m+1)r}\right) - \Lambda_0\left(\frac{jU}{(m+1)r}\right) + \frac{(2j+1)\eta r U^2}{2(m+1)^2 r^2} \right. \\
&\quad \left. - \frac{\rho U}{(m+1)r}\left(\lambda_0\left(\frac{jU}{(m+1)r}\right) + \frac{j\eta r U}{(m+1)r} - \lambda_0(0)\right)\right).
\end{aligned}
\tag{20}
$$

5. 2-D PM policy

This section adds a new dimension to a one-dimensional PM policy. In real life, PM for some capital-intensive products such as automobiles is 2-D: Two successive PM actions are separated by a time or usage interval, and the instant of the next action depends on which interval the usage process first passes through. While 2-D PM is usually offered as a uniform program to all customers, what we consider is a personalized PM service for a specific usage process or individual.

Our model assumes that a 2-D PM policy has a time interval of $T/(n+1)$ and a usage interval of $U/(m+1)$, as is the case with most car companies. The integers $n \ge 0$ and $m \ge 0$ are decision variables. This policy does not divide the 2-D warranty region into an $(n+1) \times (m+1)$ grid of cells, but generates a rectangle with length $T/(n+1)$ and width $U/(m+1)$ at each PM point to trigger the next PM action (as illustrated in Fig. 3). It is worth noting that the 2-D policy will reduce to a one-dimensional policy under a linear usage process. Specifically, PM will always be carried out in the time dimension if $0 < r \le \frac{(n+1)U}{(m+1)T}$ and in the usage dimension otherwise. Moreover, one unit of usage can translate into $1/r$ units of time, and the resulting PM schedule will be periodic. However, when assuming a gamma usage process, we face uncertainties in PM instants and the number of PM actions because of the possibility of performing PM in either dimension.

The expected total cost of the 2-D PM policy is calculated through the use of dynamic programming. Given that the product's age is x and cumulative usage is y at the kth PM instant, we define $J_k(x, y)$ as the expected total cost to the manufacturer from point (x, y) to the end of the 2-D warranty. Note that the expected cost-to-go is independent of k. We therefore omit the subscript k and use $J(x, y)$ to denote the cost-to-go function defined over the 2-D warranty region upon PM. In deriving the recursive equation for $J(x, y)$, we have four cases depending on the position of (x, y) relative to the 2-D warranty boundary, as shown in Fig. 3. As we move from one PM instant to the next, since the total number of PM actions is uncertain, we need to determine whether and how the next PM action is triggered.

In region I, i.e., when $nT/(n+1) \le x < T$ and $mU/(m+1) \le y < U$, the manufacturer will not take any PM actions but instead will just bear possible repair costs because the distance from point (x, y) to either limit of the 2-D warranty is no greater than the length of the corresponding PM interval. In this case, the value function $J(x,y)$ can be computed recursively as follows:

$$\begin{aligned} &J(x,y)\\ &= c_p + c_r \int_0^{T-x} \mathbb{E}\left[\int_x^{x+t} \left(\lambda_0(s)+\eta M(s)-\rho\left(\lambda_0(x)+\eta y-\lambda_0(0)\right)\right) \mathrm{d}s \right.\\ &\quad \left. \times \;\middle|\; M(x+t)=U\right] f_{\tau(U-y)}(t)\,\mathrm{d}t\\ &\quad + c_r \int_{T-x}^{+\infty} \mathbb{E}\left[\int_x^{T} \left(\lambda_0(s)+\eta M(s)-\rho\left(\lambda_0(x)+\eta y-\lambda_0(0)\right)\right) \mathrm{d}s \right.\\ &\quad \left. \times \;\middle|\; M(x+t)=U\right] f_{\tau(U-y)}(t)\,\mathrm{d}t\\ &= c_p + c_r \int_0^{T-x} \left(\Lambda_0(x+t)-\Lambda_0(x)+\frac{\eta(U+y)t}{2}-\rho\left(\lambda_0(x)+\eta y-\lambda_0(0)\right)t\right)\\ &\quad \times f_{\tau(U-y)}(t)\,\mathrm{d}t\\ &\quad + c_r \int_{T-x}^{+\infty} \left(\Lambda_0(T)-\Lambda_0(x)+\eta y(T-x)+\frac{\eta(U-y)(T-x)^2}{2t}\right.\\ &\quad \left. -\;\rho\left(\lambda_0(x)+\eta y-\lambda_0(0)\right)(T-x)\right) f_{\tau(U-y)}(t)\,\mathrm{d}t. \end{aligned} \tag{21}$$

To calculate the expected repair cost, we need to know whether a gamma process starting from zero hits $U-y$ for the first time before time $T-x$ or, equivalently, which limit the usage process hits first. The second term of the first equation represents the case where the usage process hits the usage limit first, whereas the third term represents the case of hitting the age limit first. Note that the condition $M(x+t)=U$ is equivalent to $\tau(U-y)=t$.

In region II, i.e., when $nT/(n+1) \le x < T$ and $0 < y < mU/(m+1)$, since the distance from point (x,y) to the age limit is no greater than the length of the time interval, the manufacturer will not perform PM based on this dimension, but the possibility for PM still remains in the other dimension. We need to determine in this region whether the usage process passes through a usage interval first before reaching the age limit. If a gamma process starting from zero hits $U/(m+1)$ for the first time before time $T-x$, then the initiated PM task entails a cost of c_p. Otherwise, the manufacturer only incurs repair costs until the end of the 2-D warranty. For any point in region II, we can express the value function $J(x,y)$ as follows:

$$\begin{aligned} &J(x,y)\\ &= c_p + c_r \int_0^{T-x} \mathbb{E}\left[\int_x^{x+t} \left(\lambda_0(s)+\eta M(s)-\rho\left(\lambda_0(x)+\eta y-\lambda_0(0)\right)\right) \mathrm{d}s \right.\\ &\quad \left. \times \;\middle|\; M(x+t)=y+\frac{U}{m+1}\right] f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}t\\ &\quad + \int_0^{T-x} J\left(x+t, y+\frac{U}{m+1}\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}t\\ &\quad + c_r \int_{T-x}^{+\infty} \mathbb{E}\left[\int_x^{T} \left(\lambda_0(s)+\eta M(s)-\rho\left(\lambda_0(x)+\eta y-\lambda_0(0)\right)\right) \mathrm{d}s \right.\\ &\quad \left. \;\middle|\; M(x+t)=y+\frac{U}{m+1}\right] f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}t\\ &= c_p + c_r \int_0^{T-x} \left(\Lambda_0(x+t)-\Lambda_0(x)+\eta y t+\frac{\eta U t}{2(m+1)}\right.\\ &\quad \left. -\;\rho\left(\lambda_0(x)+\eta y-\lambda_0(0)\right)t\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}t\\ &\quad + \int_0^{T-x} J\left(x+t, y+\frac{U}{m+1}\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}t\\ &\quad + c_r \int_{T-x}^{+\infty} \left(\Lambda_0(T)-\Lambda_0(x)+\eta y(T-x)+\frac{\eta U(T-x)^2}{2(m+1)t}\right.\\ &\quad \left. -\;\rho\left(\lambda_0(x)+\eta y-\lambda_0(0)\right)(T-x)\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}t. \end{aligned} \tag{22}$$

The second and third terms of the first equation represent the case in which the usage process first passes through a usage interval, while the fourth term corresponds to the case where the usage process first hits the age limit. Since $J(x,y)$ is an expected cost-to-go function, the third term is not multiplied by the cost parameter c_r.

In region III, i.e., when $0 < x < nT/(n+1)$ and $mU/(m+1) \le y < U$, the distance between point (x,y) and the usage limit is insufficient for PM to be performed based on usage. If a PM action is triggered, then it must be performed in the time dimension, and the problem proceeds to the next stage because the 2-D warranty is still in effect. We can write the recursive equation as

$$\begin{aligned} &J(x,y)\\ &= c_p + c_r \int_0^{\frac{T}{n+1}} \mathbb{E}\left[\int_x^{x+t} \left(\lambda_0(s)+\eta M(s)-\rho\left(\lambda_0(x)+\eta y-\lambda_0(0)\right)\right) \mathrm{d}s \right.\\ &\quad \left. \times \;\middle|\; M(x+t)=U\right] f_{\tau(U-y)}(t)\,\mathrm{d}t\\ &\quad + c_r \int_{\frac{T}{n+1}}^{+\infty} \mathbb{E}\left[\int_x^{x+\frac{T}{n+1}} \left(\lambda_0(s)+\eta M(s)-\rho\left(\lambda_0(x)+\eta y-\lambda_0(0)\right)\right) \mathrm{d}s \right.\\ &\quad \left. \times \;\middle|\; M(x+t)=U\right] f_{\tau(U-y)}(t)\,\mathrm{d}t\\ &\quad + \int_{\frac{T}{n+1}}^{+\infty} \mathbb{E}\left[J\left(x+\frac{T}{n+1}, M\left(x+\frac{T}{n+1}\right)\right)\right.\\ &\quad \left. \times \;\middle|\; M(x+t)=U\right] f_{\tau(U-y)}(t)\,\mathrm{d}t\\ &= c_p + c_r \int_0^{\frac{T}{n+1}} \left(\Lambda_0(x+t)-\Lambda_0(x)+\frac{\eta(U+y)t}{2}-\rho\left(\lambda_0(x)+\eta y-\lambda_0(0)\right)t\right)\\ &\quad \times f_{\tau(U-y)}(t)\,\mathrm{d}t\\ &\quad + c_r \int_{\frac{T}{n+1}}^{+\infty} \left(\Lambda_0\left(x+\frac{T}{n+1}\right)-\Lambda_0(x)+\frac{\eta y T}{n+1}+\frac{\eta(U-y)T^2}{2(n+1)^2 t}\right.\\ &\quad \left. -\frac{\rho\left(\lambda_0(x)+\eta y-\lambda_0(0)\right)T}{n+1}\right) f_{\tau(U-y)}(t)\,\mathrm{d}t\\ &\quad + \int_{\frac{T}{n+1}}^{+\infty} \mathbb{E}\left[J\left(x+\frac{T}{n+1}, M\left(x+\frac{T}{n+1}\right)\right) \;\middle|\; M(x+t)=U\right]\\ &\quad \times f_{\tau(U-y)}(t)\,\mathrm{d}t. \end{aligned} \tag{23}$$

Note that $U-y$ is the distance between point (x,y) and the usage limit. The second term of the first equation indicates that the time required for the usage path to hit $U-y$ for the first time is less than the length of the time interval, so the 2-D warranty ends before the next PM instant. The third and fourth terms indicate that a PM action occurs at time $x+T/(n+1)$. The second argument of $J(x+T/(n+1), M(x+T/(n+1)))$ in the fourth term still involves uncertainty, unlike that of $J(x+t, y+U/(m+1))$ in Eq. (22). We do not observe the cumulative usage at time $x+T/(n+1)$, but know that the cumulative usage at time $x+t$ is U. Given a gamma bridge that begins at $M(x)=y$ and terminates at $M(x+t)=U$, the value of the bridge at time $x+T/(n+1)$, $M(x+T/(n+1))$, is equal to y plus $U-y$ multiplied by a beta random variable with distribution $\mathrm{Beta}(\alpha T/(n+1), \alpha t-\alpha T/(n+1))$.

In region IV, i.e. when $0 < x < nT/(n+1)$ and $0 < y < mU/(m+1)$, PM is possible along both the time and usage dimensions. In this case, the recursive equation is written as

$$\begin{aligned}
&J(x,y)\\
&= c_p + c_r\int_0^{\frac{T}{n+1}} \mathbb{E}\left[\int_x^{x+t}\left(\lambda_0(s)+\eta M(s)-\rho\big(\lambda_0(x)+\eta y-\lambda_0(0)\big)\right)\mathrm{d}\,s\right.\\
&\quad\times\left.\left|\, M(x+t)=y+\frac{U}{m+1}\right.\right] f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t\\
&\quad+\int_0^{\frac{T}{n+1}} J\left(x+t, y+\frac{U}{m+1}\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t\\
&\quad+c_r\int_{\frac{T}{n+1}}^{+\infty}\mathbb{E}\left[\int_x^{x+\frac{T}{n+1}}\left(\lambda_0(s)+\eta M(s)-\rho\big(\lambda_0(x)+\eta y-\lambda_0(0)\big)\right)\mathrm{d}\,s\right.\\
&\quad\left.\left|\, M(x+t)=y+\frac{U}{m+1}\right.\right] f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t\\
&\quad+\int_{\frac{T}{n+1}}^{+\infty}\mathbb{E}\left[J\left(x+\frac{T}{n+1}, M\left(x+\frac{T}{n+1}\right)\right)\,\middle|\, M(x+t)=y+\frac{U}{m+1}\right]\\
&\quad\cdot f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t\\
&= c_p + c_r\int_0^{\frac{T}{n+1}}\left(\Lambda_0(x+t)-\Lambda_0(x)+\eta yt+\frac{\eta Ut}{2(m+1)}\right.\\
&\quad\left.-\rho\big(\lambda_0(x)+\eta y-\lambda_0(0)\big)t\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t\\
&\quad+\int_0^{\frac{T}{n+1}} J\left(x+t, y+\frac{U}{m+1}\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t\\
&\quad+c_r\int_{\frac{T}{n+1}}^{+\infty}\left(\Lambda_0\left(x+\frac{T}{n+1}\right)-\Lambda_0(x)+\frac{\eta yT}{n+1}+\frac{\eta UT^2}{2(m+1)(n+1)^2 t}\right.\\
&\quad\left.-\frac{\rho\big(\lambda_0(x)+\eta y-\lambda_0(0)\big)T}{n+1}\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t\\
&\quad+\int_{\frac{T}{n+1}}^{+\infty}\mathbb{E}\left[J\left(x+\frac{T}{n+1}, M\left(x+\frac{T}{n+1}\right)\right)\,\middle|\, M(x+t)=y+\frac{U}{m+1}\right]\\
&\quad\cdot f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t.
\end{aligned} \tag{24}$$

We can determine whether the usage process first traverses a time or usage interval by comparing the first hitting time of $U/(m+1)$ against $T/(n+1)$. If the former is smaller than the latter, then the next PM action will take place at point $(x+t, y+U/(m+1))$, as captured by the second and third terms of the first equation. Otherwise, the next PM action will occur at point $(x+T/(n+1), M(x+T/(n+1)))$, as indicated by the fourth and fifth terms. The cumulative usage $M(x+T/(n+1))$ is equal to y plus $U/(m+1)$ multiplied by a beta random variable with distribution Beta$(\alpha T/(n+1), \alpha t-\alpha T/(n+1))$. Finally, the boundary conditions of the dynamic program are given by $J(T,\cdot)=0$ and $J(\cdot,U)=0$.

Our ultimate goal is to find the expected total cost of the 2-D PM policy, which we denote by $C_3(n,m)$. For a gamma usage process that starts at point $(0,0)$ and ends at the warranty boundary, we use the value function $J(x,y)$ to express $C_3(n,m)$ as

$$\begin{aligned}
&C_3(n,m)\\
&= c_r\int_0^{\frac{T}{n+1}}\mathbb{E}\left[\int_0^t\big(\lambda_0(s)+\eta M(s)\big)\mathrm{d}\,s\,\middle|\, M(t)=\frac{U}{m+1}\right] f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t\\
&\quad+\int_0^{\frac{T}{n+1}} J\left(t,\frac{U}{m+1}\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t\\
&\quad+c_r\int_{\frac{T}{n+1}}^{+\infty}\mathbb{E}\left[\int_0^{\frac{T}{n+1}}\big(\lambda_0(s)+\eta M(s)\big)\mathrm{d}\,s\,\middle|\, M(t)=\frac{U}{m+1}\right] f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t\\
&\quad+\int_{\frac{T}{n+1}}^{+\infty}\mathbb{E}\left[J\left(\frac{T}{n+1}, M\left(\frac{T}{n+1}\right)\right)\right.\\
&\quad\left.\left|\, M(t)=\frac{U}{m+1}\right.\right] f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t\\
&= c_r\int_0^{\frac{T}{n+1}}\left(\Lambda_0(t)+\frac{\eta Ut}{2(m+1)}\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t\\
&\quad+\int_0^{\frac{T}{n+1}} J\left(t,\frac{U}{m+1}\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t + c_r\int_{\frac{T}{n+1}}^{+\infty}\left(\Lambda_0\left(\frac{T}{n+1}\right)\right.\\
&\quad\left.+\frac{\eta UT^2}{2(m+1)(n+1)^2 t}\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t + \int_{\frac{T}{n+1}}^{+\infty}\mathbb{E}\left[J\left(\frac{T}{n+1}, M\left(\frac{T}{n+1}\right)\right)\right.\\
&\quad\times\left.\left|\, M(t)=\frac{U}{m+1}\right.\right] f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\,t.
\end{aligned} \tag{25}$$

In the fourth term of the first equation, $M(T/(n+1))$ is equal to $U/(m+1)$ times a beta random variable with distribution Beta$(\alpha T/(n+1), \alpha t-\alpha T/(n+1))$. It is worth noting that PM is not performed at point $(0,0)$, and hence the manufacturer does not start out by incurring a cost of c_p. After discretizing the 2-D warranty region, we compute the values of $J(x,y)$ in a recursive manner. Then, for given nonnegative integers n and m, we calculate the expected total cost $C_3(n,m)$ from Eq. (25). Finally, we find the optimal 2-D PM policy (n^*, m^*) via a grid search over the parameter space.

We can also use dynamic programming to obtain the expected total costs of the two one-dimensional PM policies studied in the previous section. When PM is time-based, we let $m=0$, in which case the usage interval of 2-D PM is U. Since we do not perform PM at the end of the 2-D warranty, the usage dimension does not come into play. Given $m=0$, only the conditions in the first and third cases for $J(x,y)$ hold. Based on this value function, an alternative way of evaluating $C_1(n)$ is presented below:

$$\begin{aligned}
&C_1(n)\\
&= c_r\int_0^{\frac{T}{n+1}}\mathbb{E}\left[\int_0^t\big(\lambda_0(s)+\eta M(s)\big)\mathrm{d}\,s\,\middle|\, M(t)=U\right] f_{\tau(U)}(t)\,\mathrm{d}\,t\\
&\quad+c_r\int_{\frac{T}{n+1}}^{+\infty}\mathbb{E}\left[\int_0^{\frac{T}{n+1}}\big(\lambda_0(s)+\eta M(s)\big)\mathrm{d}\,s\,\middle|\, M(t)=U\right]\\
&\quad\cdot f_{\tau(U)}(t)\,\mathrm{d}\,t + \int_{\frac{T}{n+1}}^{+\infty}\mathbb{E}\left[J\left(\frac{T}{n+1}, M\left(\frac{T}{n+1}\right)\right)\,\middle|\, M(t)=U\right] f_{\tau(U)}(t)\,\mathrm{d}\,t\\
&= c_r\int_0^{\frac{T}{n+1}}\left(\Lambda_0(t)+\frac{\eta Ut}{2}\right) f_{\tau(U)}(t)\,\mathrm{d}\,t\\
&\quad+c_r\int_{\frac{T}{n+1}}^{+\infty}\left(\Lambda_0\left(\frac{T}{n+1}\right)+\frac{\eta UT^2}{2(n+1)^2 t}\right) f_{\tau(U)}(t)\,\mathrm{d}\,t\\
&\quad+\int_{\frac{T}{n+1}}^{+\infty}\mathbb{E}\left[J\left(\frac{T}{n+1}, M\left(\frac{T}{n+1}\right)\right)\,\middle|\, M(t)=U\right] f_{\tau(U)}(t)\,\mathrm{d}\,t.
\end{aligned} \tag{26}$$

In the third term of the first equation, $M(T/(n+1))$ is equal to U multiplied by a beta random variable with distribution Beta$(\alpha T/(n+1), \alpha t-\alpha T/(n+1))$. Under the usage-based PM policy, we let $n=0$. Of the four cases for $J(x,y)$, only the first and second apply. Since Eq. (A.2) is challenging to compute when $m \geq 2$, we proceed as follows to evaluate

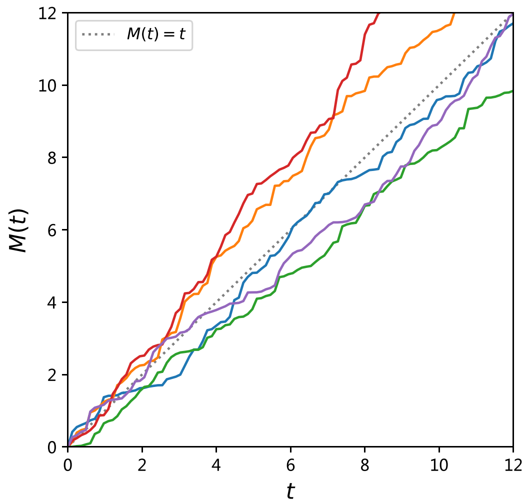

Fig. 4. Sample paths of the gamma usage process ($\alpha = \beta = 8.333$).

$C_2(m)$:

$$\begin{aligned} &C_2(m)\\ &= c_r \int_0^T \mathbb{E}\left[\int_0^t \left(\lambda_0(s) + \eta M(s)\right) \mathrm{d}\, s \,\middle|\, M(t) = \frac{U}{m+1}\right] f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\, t\\ &+ \int_0^T J\left(t, \frac{U}{m+1}\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\, t\\ &+ c_r \int_T^{+\infty} \mathbb{E}\left[\int_0^T \left(\lambda_0(s) + \eta M(s)\right) \mathrm{d}\, s \,\middle|\, M(t) = \frac{U}{m+1}\right] f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\, t\\ &= c_r \int_0^T \left(\Lambda_0(t) + \frac{\eta U t}{2(m+1)}\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\, t\\ &+ \int_0^T J\left(t, \frac{U}{m+1}\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\, t\\ &+ c_r \int_T^{+\infty} \left(\Lambda_0(T) + \frac{\eta U T^2}{2(m+1)t}\right) f_{\tau\left(\frac{U}{m+1}\right)}(t)\,\mathrm{d}\, t. \end{aligned} \tag{27}$$

It is straightforward to verify that $C_1(n) = C_3(n, 0)$ and $C_2(m) = C_3(0, m)$. Thus, the time-based and usage-based PM policies are special cases of the 2-D PM policy.

6. Numerical study

In this section, we present the results of a numerical study to illustrate how the variability of a usage process affects the performance of the three PM policies. We use the following values for the model parameters: $T = 12$, $U = 12$, $\lambda_0(t) = 0.05$, $\eta = 0.1$, $\rho = 0.9$, $c_p = 100$, and $c_r = 300$. To measure the process variability, we choose the coefficient of variation of $M(T)$, defined by $CV = \sqrt{\mathrm{Var}[M(T)]}/\mathbb{E}[M(T)]$. The underlying assumption is that as the level of use increases, the range of fluctuation gets broader, and the occurrence of larger jumps becomes more likely. Since $M(T) \sim \mathrm{Gamma}(\alpha T, \beta)$, given the values of $\mathbb{E}[M(T)]$ and CV, we have $\alpha = 1/(CV^2 T)$ and $\beta = 1/(CV^2 \mathbb{E}[M(T)])$. Specifically, we let $CV = 0.1$ and $\mathbb{E}[M(T)] = rT$. Fig. 4 plots five sample paths of the gamma usage process with $\mathbb{E}[M(T)] = 12$. Note that only one sample path can be observed in real-world scenarios. Additionally, we find that calculating the usage rate r using the usage at the last failure may lead to significant errors.

To explore the effect of the variability, we consider for illustration the cases $n = 3$ and $m = 3$. When there is no randomness (i.e., $CV = 0$), we have $M(t) = rt$. The expected total cost of the time-based PM policy can be obtained from Eq. (15) and is depicted as a function of r in Fig. 5(a). For $CV = 0$, the cost curve first increases due to the effect of usage and then decreases because the 2-D warranty ends earlier. Beyond a certain point, this curve increases again because the last PM action becomes undesirable. The number of PM actions is reduced by one at $r = 1.333$, causing a sharp drop in the cost. For $CV = 0.1$, we see that the variability leads to a smooth cost curve. The usage rate and warranty end time continue to be the main influencing factors. Fig. 5(b) presents the percentage error between the costs in these two cases and demonstrates the existence of both overestimation and underestimation errors. In Fig. 6, a similar analysis can be done for the expected total cost of the usage-based PM policy. Note that when $m \geq 3$, we need to determine this cost by solving the dynamic program because it is impractical to compute the high-dimensional integrals in Eq. (A.2).

Fig. 7 demonstrates the impact of the process variability on PM schedules under the 2-D PM policy. Recall that without any variability, the policy reduces to a one-dimensional policy, as represented by the dotted lines in the two subplots. When the randomness is present, we calculate the average number of PM actions and the percentage of usage-based PM actions over 10,000 usage paths. The left plot shows that the closer the usage rate is to $\frac{(n+1)U}{(m+1)T}$, the higher the average number of PM actions by virtue of an additional PM point near the 2-D warranty boundary. As seen in the right plot, another effect of the randomness is that the closer the usage rate is to $\frac{(n+1)U}{(m+1)T}$, the more likely it is that PM will be performed based on the other dimension. Fig. 8 displays the expected total cost of the 2-D PM policy. For small and large values of r, since PM schedules are almost one-dimensional, the percentage error in Fig. 8(b) should be roughly equal to that in Figs. 5(b) and 6(b), respectively. For medium r, performing one more PM action generally increases the manufacturer's expected total cost, and the percentage error depends on the benefit of this action.

Fig. 9 compares the optimal costs of the two one-dimensional PM policies in the absence of the randomness. In Fig. 9(a), when $0 < r \leq U/T$, both curves increase with the usage rate, and the time-based policy is less costly than the usage-based policy; however, when $r > U/T$, the curves are both decreasing and the comparison result is reversed. When they overlap, the PM schedules resulting from the two one-dimensional policies are the same. The minimum of the two costs is the optimal cost of the 2-D PM policy. Fig. 10 shows the cost difference when the randomness is present. It is interesting to observe that the process variability does not change the comparison result. Although a one-dimensional policy is a special case of a 2-D policy, for almost all values of r, the optimal 2-D policy is degenerate, indicating that there is little cost benefit to be gained from performing 2-D PM as a result of additional PM points near the warranty boundary. However, the situation may be different when the manufacturer only bears a portion of the PM cost through cost-sharing programs, and the customer is responsible for the remaining PM visits during the warranty period [50]. 2-D PM may also benefit the manufacturer when product usage is highly variable or when uncertainty regarding customer types arises. These are some of the reasons why 2-D PM is so prevalent in the automotive industry.

To compare uniform and personalized 2-D PM policies, we examine the manufacturer's expected total cost at the population level, as detailed in Table 2. We first normalize the size of the customer population to one and then consider the following two usage distributions: the first is $r \sim U[0.5, 1.5]$ with $\mathbb{E}[r] = 1$ and $\mathrm{Var}[r] = 0.083$; the second is $\log r \sim N(0.0984, 0.58^2)$ with $\mathbb{E}[r] = 1.21$ and $\mathrm{Var}[r] = 0.586$. For uniform PM, we first compute the population cost for each (n, m) and then minimize over (n, m), while for personalized PM, we first minimize over (n, m) for each r and then compute the population cost. When $CV = 0$, the optimal uniform policies under the two distributions are both $(n^*, m^*) = (3, 3)$. The more dispersed the usage rate distribution, the larger the percentage cost reduction. When the randomness exists, we have $(n^*, m^*) = (2, 3)$ and $(3, 2)$, respectively. More benefits are achieved because uniform PM does not allow the degeneration of the optimal 2-D policy. We also observe that the assumption of constant usage rates leads to underestimation errors.

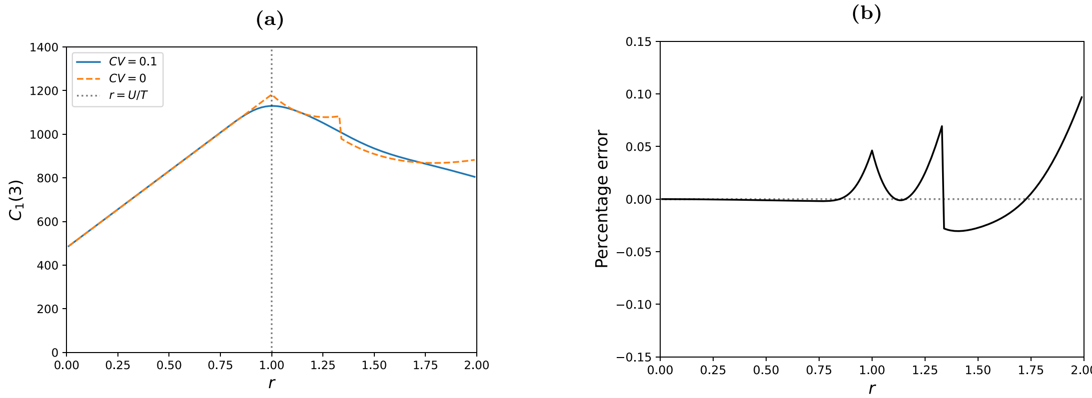

Fig. 5. Effect of process variation on $C_1(n)$ when $n = 3$.

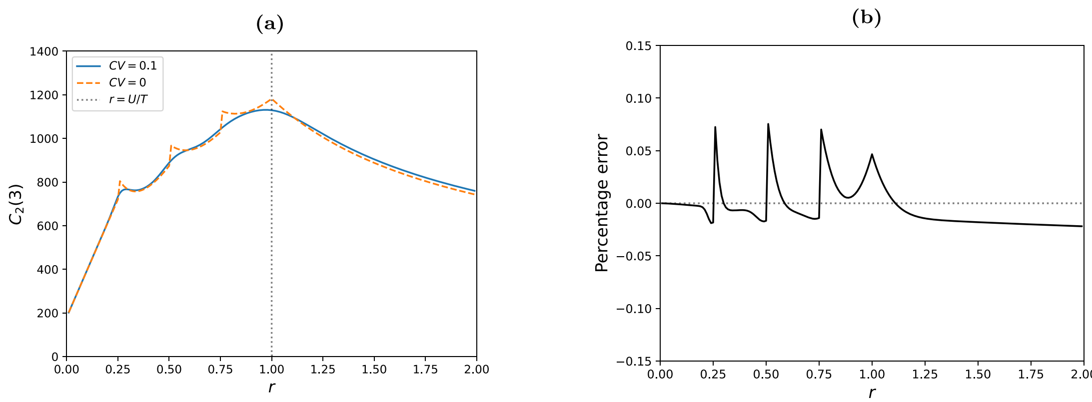

Fig. 6. Effect of process variation on $C_2(m)$ when $m = 3$.

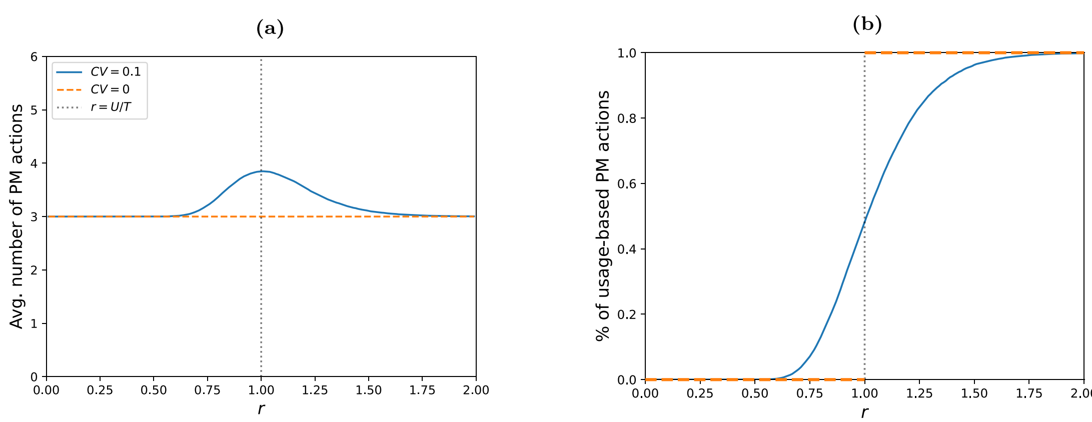

Fig. 7. Effect of process variation on PM schedules when $n = 3$ and $m = 3$ (10,000 sample paths).

Table 2
Manufacturer's costs for uniform and personalized 2-D PM under two usage rate distributions.

	CV	Uniform PM	Personalized PM	Cost reduction
$r \sim U[0.5, 1.5]$	0	1010.862	1009.748	0.11%
	0.1	1023.089	1010.504	1.25%
$\log r \sim N(0.0984, 0.58^2)$	0	907.897	896.524	1.23%
	0.1	915.335	900.281	1.64%

7. Conclusion

In this paper, we consider a manufacturer adopting sensor technology to continuously monitor the usage process of a product that is sold with a 2-D warranty. To account for the random nature of usage, we assume that the cumulative usage of the product follows a gamma process. Using the concepts of first hitting times, gamma bridges, and doubly stochastic Poisson processes, we formulate a dynamic programming model to determine the expected total costs of three PM policies under the 2-D warranty. Through numerical experiments, we show how their expected total costs are affected by the randomness of the usage process. We also find that the optimal 2-D PM policy degenerates into a time- or usage-based policy when the manufacturer incurs all PM costs.

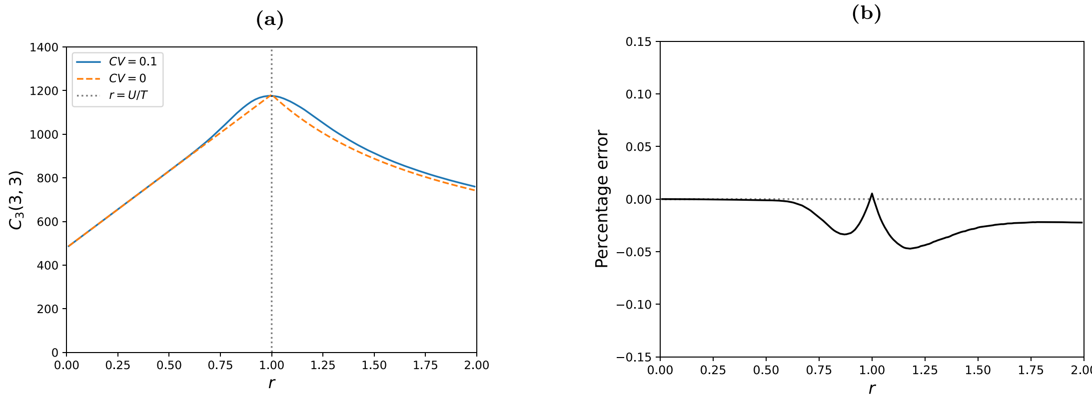

Fig. 8. Effect of process variation on $C_3(n, m)$ when $n = 3$ and $m = 3$.

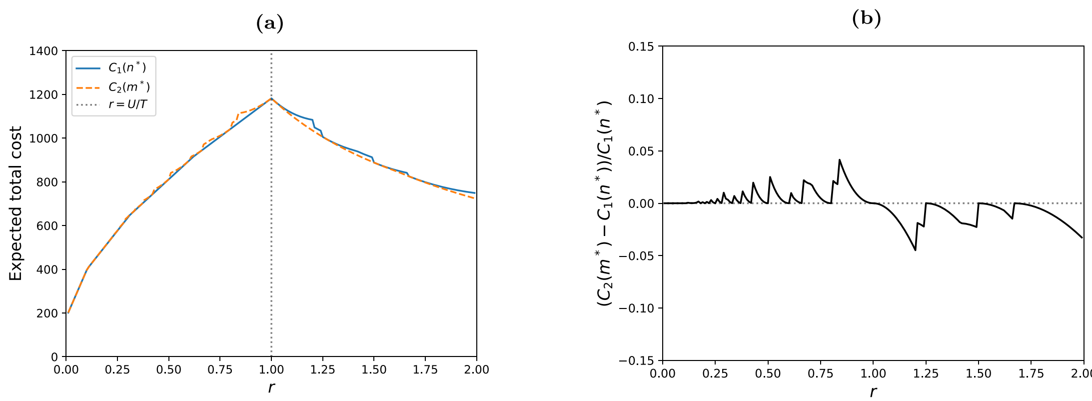

Fig. 9. Comparison between the optimal costs of the PM policies ($CV = 0$).

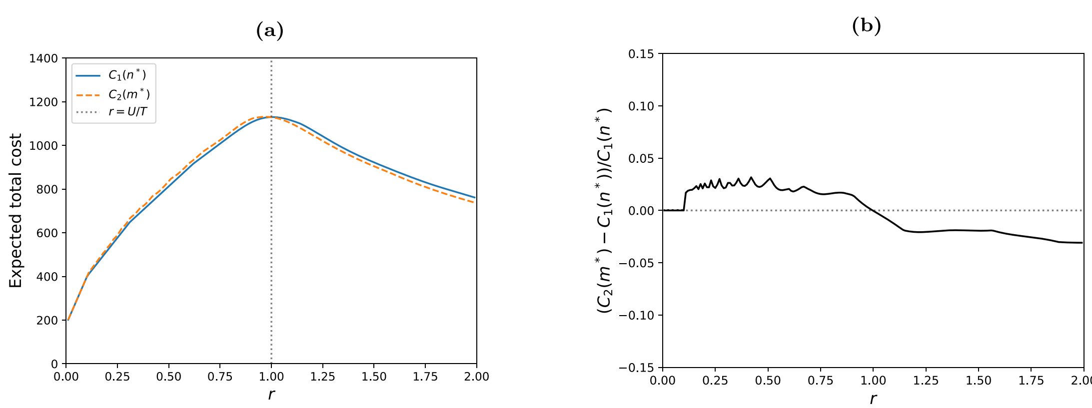

Fig. 10. Comparison between the optimal costs of the PM policies ($CV = 0.1$).

Our proposed model has some limitations that suggest directions for future research. The first issue relates to the validity of the gamma process assumption. A possible extension would be to test it on real-world data by checking whether the shape function is linear and whether the hazards model is additive. Usage paths tend to vary significantly across product units. To reflect the heterogeneity in usage, a gamma process model with random effects needs to be developed [51].

Second, we derive the optimal PM policies for given parameters of a gamma usage process. These parameters, however, may be unknown in advance. For estimation purposes, manufacturers can use sensors to track a product's usage process over a period of time before the first PM action is triggered. When usage uncertainty unfolds over time, investigating dynamic PM policies is a promising research direction. Such policies require manufacturers to periodically update the process parameters (e.g., in a Bayesian framework), resolve the problem, and adjust PM schedules accordingly.

Third, when the cumulative usage at failure is strongly correlated with age, a unidimensional failure rate function with respect to time is

$$
\begin{aligned}
&\mathbb{E}\Bigg[\mathbb{E}\Bigg[\text{repair cost}\ \Bigg|\ \tau_1\left(\frac{U}{m+1}\right),\ldots,\tau_{m+1}\left(\frac{U}{m+1}\right)\Bigg]\Bigg]\\
&= c_r\int_T^{\infty}\mathbb{E}\Bigg[\int_0^T\lambda(s)\,\mathrm{d}s\ \Bigg|\ \tau_1=t_1\Bigg]f_{\tau_1}(t_1)\,\mathrm{d}t_1+c_r\sum_{i=1}^{m}\int\cdots\int_{\sum_{j=1}^{i}t_j<T\le\sum_{j=1}^{i+1}t_j}\mathbb{E}\Bigg[\int_0^{t_1}\lambda(s)\,\mathrm{d}s+\sum_{j=1}^{i-1}\int_{\sum_{h=1}^{j}t_h}^{\sum_{h=1}^{j+1}t_h}\lambda(s)\,\mathrm{d}s\\
&\quad+\int_{\sum_{h=1}^{i}t_h}^{T}\lambda(s)\,\mathrm{d}s\ \Bigg|\ \tau_1=t_1,\ldots,\tau_{i+1}=t_{i+1}\Bigg]f_{\tau_1}(t_1)\ldots f_{\tau_{i+1}}(t_{i+1})\,\mathrm{d}t_{i+1}\ldots\mathrm{d}t_1+c_r\int\cdots\int_{\sum_{j=1}^{m+1}t_j<T}\\
&\quad\mathbb{E}\Bigg[\int_0^{t_1}\lambda(s)\,\mathrm{d}s+\sum_{j=1}^{m}\int_{\sum_{h=1}^{j}t_h}^{\sum_{h=1}^{j+1}t_h}\lambda(s)\,\mathrm{d}s\ \Bigg|\ \tau_1=t_1,\ldots,\tau_{m+1}=t_{m+1}\Bigg]f_{\tau_1}(t_1)\ldots f_{\tau_{m+1}}(t_{m+1})\,\mathrm{d}t_{m+1}\ldots\mathrm{d}t_1\\
&= c_r\int_T^{\infty}\int_0^T\mathbb{E}\Bigg[\lambda_0(s)+\eta M(s)\ \Bigg|\ M(t_1)=\frac{U}{m+1}\Bigg]\mathrm{d}s\,f_{\tau_1}(t_1)\,\mathrm{d}t_1+c_r\sum_{i=1}^{m}\int_0^T\int_0^{T-t_1}\cdots\int_0^{T-\sum_{j=1}^{i-1}t_j}\int_{T-\sum_{j=1}^{i}t_j}^{\infty}\\
&\quad\Bigg\{\int_0^{t_1}\mathbb{E}\Bigg[\lambda_0(s)+\eta M(s)\ \Bigg|\ M(t_1)=\frac{U}{m+1}\Bigg]\mathrm{d}s+\sum_{j=1}^{i-1}\int_{\sum_{h=1}^{j}t_h}^{\sum_{h=1}^{j+1}t_h}\mathbb{E}\Bigg[\lambda_0(s)+\eta M(s)-\rho\Bigg(\lambda_0\Bigg(\sum_{h=1}^{j}t_h\Bigg)\\
&\quad+\eta M\Bigg(\sum_{h=1}^{j}t_h\Bigg)-\lambda_0(0)\Bigg)\ \Bigg|\ M\Bigg(\sum_{h=1}^{j}t_h\Bigg)=\frac{jU}{m+1},M\Bigg(\sum_{h=1}^{j+1}t_h\Bigg)=\frac{(j+1)U}{m+1}\Bigg]\mathrm{d}s+\int_{\sum_{h=1}^{i}t_h}^{T}\mathbb{E}\Bigg[\lambda_0(s)+\eta M(s)\\
&\quad-\rho\Bigg(\lambda_0\Bigg(\sum_{h=1}^{i}t_h\Bigg)+\eta M\Bigg(\sum_{h=1}^{i}t_h\Bigg)-\lambda_0(0)\Bigg)\ \Bigg|\ M\Bigg(\sum_{h=1}^{i}t_h\Bigg)=\frac{iU}{m+1},M\Bigg(\sum_{h=1}^{i+1}t_h\Bigg)=\frac{(i+1)U}{m+1}\Bigg]\mathrm{d}s\Bigg\}\\
&\quad\cdot f_{\tau_1}(t_1)\ldots f_{\tau_{i+1}}(t_{i+1})\,\mathrm{d}t_{i+1}\ldots\mathrm{d}t_1+c_r\int_0^T\int_0^{T-t_1}\cdots\int_0^{T-\sum_{j=1}^{m}t_j}\Bigg\{\int_0^{t_1}\mathbb{E}\Bigg[\lambda_0(s)+\eta M(s)\ \Bigg|\ M(t_1)=\frac{U}{m+1}\Bigg]\mathrm{d}s\\
&\quad+\sum_{j=1}^{m}\int_{\sum_{h=1}^{j}t_h}^{\sum_{h=1}^{j+1}t_h}\mathbb{E}\Bigg[\lambda_0(s)+\eta M(s)-\rho\Bigg(\lambda_0\Bigg(\sum_{h=1}^{j}t_h\Bigg)+\eta M\Bigg(\sum_{h=1}^{j}t_h\Bigg)-\lambda_0(0)\Bigg)\ \Bigg|\ M\Bigg(\sum_{h=1}^{j}t_h\Bigg)=\frac{jU}{m+1},\\
&\quad M\Bigg(\sum_{h=1}^{j+1}t_h\Bigg)=\frac{(j+1)U}{m+1}\Bigg]\mathrm{d}s\Bigg\}f_{\tau_1}(t_1)\ldots f_{\tau_{m+1}}(t_{m+1})\,\mathrm{d}t_{m+1}\ldots\mathrm{d}t_1\\
&= c_r\int_T^{\infty}\left(\Lambda_0(T)+\frac{\eta UT^2}{2(m+1)t_1}\right)f_{\tau_1}(t_1)\,\mathrm{d}t_1+c_r\sum_{i=1}^{m}\int_0^T\int_0^{T-t_1}\cdots\int_0^{T-\sum_{j=1}^{i-1}t_j}\int_{T-\sum_{j=1}^{i}t_j}^{\infty}\Bigg\{\Lambda_0(T)+\frac{\eta Ut_1}{2(m+1)}\\
&\quad+\frac{(1-\rho)\eta U\sum_{j=1}^{i-1}jt_{j+1}}{m+1}+\frac{\eta U\sum_{j=1}^{i-1}t_{j+1}}{2(m+1)}-\rho\sum_{j=1}^{i-1}t_{j+1}\lambda_0\Bigg(\sum_{h=1}^{j}t_h\Bigg)+\frac{i\eta U(T-\sum_{h=1}^{i}t_h)}{m+1}+\frac{\eta U\left(T^2+(\sum_{h=1}^{i}t_h)^2\right)}{2(m+1)t_{i+1}}\\
&\quad-\frac{\eta UT\sum_{h=1}^{i}t_h}{(m+1)t_{i+1}}-\rho\Bigg(\lambda_0\Bigg(\sum_{h=1}^{i}t_h\Bigg)+\frac{i\eta U}{m+1}\Bigg)\Bigg(T-\sum_{h=1}^{i}t_h\Bigg)+\rho\lambda_0(0)(T-t_1)\Bigg\}f_{\tau_1}(t_1)\ldots f_{\tau_{i+1}}(t_{i+1})\,\mathrm{d}t_{i+1}\ldots\mathrm{d}t_1\\
&\quad+c_r\int_0^T\int_0^{T-t_1}\cdots\int_0^{T-\sum_{j=1}^{m}t_j}\Bigg(\Lambda_0\Bigg(\sum_{h=1}^{m+1}t_h\Bigg)+\frac{\eta Ut_1}{2(m+1)}+\frac{(1-\rho)\eta U\sum_{j=1}^{m}jt_{j+1}}{m+1}+\frac{\eta U\sum_{j=1}^{m}t_{j+1}}{2(m+1)}\\
&\quad-\rho\sum_{j=1}^{m}t_{j+1}\lambda_0\Bigg(\sum_{h=1}^{j}t_h\Bigg)+\rho\lambda_0(0)\sum_{j=1}^{m}t_{j+1}\Bigg)f_{\tau_1}(t_1)\ldots f_{\tau_{m+1}}(t_{m+1})\,\mathrm{d}t_{m+1}\ldots\mathrm{d}t_1.
\end{aligned}
\tag{A.2}
$$

Box I.

an adequate model to use. However, for the weakly correlated case, it is better to consider a distribution of time and usage to failure before implementing a 2-D PM policy. After obtaining a 2-D failure distribution under a gamma usage process, one could apply 2-D renewal theory to the study of preventive replacement for nonrepairable products.

CRediT authorship contribution statement

Shizhe Peng: Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Wei Jiang:** Writing – review & editing, Supervision, Funding acquisition, Conceptualization. **Wenpo Huang:** Writing – review & editing, Project administration, Funding acquisition, Conceptualization, Validation. **Qinglin Luo:** Software, Investigation, Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix

In this appendix, we derive explicit expressions for the expected repair costs of the two one-dimensional PM policies. The expected repair cost in Eq. (15) can be further written as follows:

$$
\begin{aligned}
& c_r\mathbb{E}\left[\sum_{j=0}^{\lceil (n+1)\hat{T}/T\rceil -2} \int_{\frac{jT}{n+1}}^{\frac{(j+1)T}{n+1}} \lambda(s)\,\mathrm{d}s + \int_{\frac{\left(\lceil (n+1)\hat{T}/T\rceil -1\right)T}{n+1}}^{\hat{T}} \lambda(s)\,\mathrm{d}s\right] \\
&= c_r \int_0^{\infty} \mathbb{E}\left[\sum_{j=0}^{\lceil (n+1)\hat{T}/T\rceil -2} \int_{\frac{jT}{n+1}}^{\frac{(j+1)T}{n+1}} \lambda(s)\,\mathrm{d}s \right.\\
&\quad \left. + \int_{\frac{\left(\lceil (n+1)\hat{T}/T\rceil -1\right)T}{n+1}}^{\hat{T}} \lambda(s)\,\mathrm{d}s \,\middle|\, \tau(U)=t\right] f_{\tau(U)}(t)\,\mathrm{d}t \\
&= c_r \sum_{i=0}^{n} \int_{\frac{iT}{n+1}}^{\frac{(i+1)T}{n+1}} \mathbb{E}\left[\sum_{j=0}^{i-1} \int_{\frac{jT}{n+1}}^{\frac{(j+1)T}{n+1}} \lambda(s)\,\mathrm{d}s + \int_{\frac{iT}{n+1}}^{t} \lambda(s)\,\mathrm{d}s \right.\\
&\quad \times \left. \,\middle|\, \tau(U)=t\right] f_{\tau(U)}(t)\,\mathrm{d}t + c_r \int_T^{\infty} \mathbb{E}\left[\sum_{j=0}^{n} \int_{\frac{jT}{n+1}}^{\frac{(j+1)T}{n+1}} \lambda(s)\,\mathrm{d}s \right.\\
&\quad \left. \,\middle|\, \tau(U)=t\right] f_{\tau(U)}(t)\,\mathrm{d}t \\
&= c_r \sum_{i=0}^{n} \int_{\frac{iT}{n+1}}^{\frac{(i+1)T}{n+1}} \left(\sum_{j=0}^{i-1} \int_{\frac{jT}{n+1}}^{\frac{(j+1)T}{n+1}} \mathbb{E}\left[\lambda_0(s)+\eta M(s) - \rho\left(\lambda_0\left(\frac{jT}{n+1}\right)\right.\right.\right.\\
&\quad \left.\left. +\eta M\left(\frac{jT}{n+1}\right) - \lambda_0(0)\right) \,\middle|\, M(t)=U\right] \mathrm{d}s \\
&\quad + \int_{\frac{iT}{n+1}}^{t} \mathbb{E}\left[\lambda_0(s)+\eta M(s) - \rho\left(\lambda_0\left(\frac{iT}{n+1}\right) + \eta M\left(\frac{iT}{n+1}\right) - \lambda_0(0)\right)\right.\\
&\quad \times \left.\left. \,\middle|\, M(t)=U\right] \mathrm{d}s\right) f_{\tau(U)}(t)\,\mathrm{d}t \\
&\quad + c_r \int_T^{\infty} \left(\sum_{j=0}^{n} \int_{\frac{jT}{n+1}}^{\frac{(j+1)T}{n+1}} \mathbb{E}\left[\lambda_0(s)+\eta M(s) - \rho\left(\lambda_0\left(\frac{jT}{n+1}\right)\right.\right.\right.\\
&\quad \left.\left.\left. + \eta M\left(\frac{jT}{n+1}\right) - \lambda_0(0)\right) \,\middle|\, M(t)=U\right] \mathrm{d}s\right) f_{\tau(U)}(t)\,\mathrm{d}t \\
&= c_r \sum_{i=0}^{n} \int_{\frac{iT}{n+1}}^{\frac{(i+1)T}{n+1}} \left(\Lambda_0(t) + \frac{\eta U t}{2} + \frac{i(i+1)\rho\eta U T^2}{2(n+1)^2 t} - \frac{i\rho\eta U T}{n+1}\right.\\
&\quad - \frac{\rho T}{n+1}\sum_{j=0}^{i-1}\lambda_0\left(\frac{jT}{n+1}\right) - \rho\left(t-\frac{iT}{n+1}\right)\lambda_0\left(\frac{iT}{n+1}\right)\\
&\quad \left. +\rho t\lambda_0(0)\right) f_{\tau(U)}(t)\,\mathrm{d}t + c_r\int_T^{\infty}\left(\Lambda_0(T) + \frac{\eta U T^2}{2t}\right.\\
&\quad \left. - \frac{\rho T}{n+1}\sum_{j=0}^{n}\lambda_0\left(\frac{jT}{n+1}\right) - \frac{n\rho\eta U T^2}{2(n+1)t} + \rho T\lambda_0(0)\right) f_{\tau(U)}(t)\,\mathrm{d}t.
\end{aligned}
\tag{A.1}
$$

In the first equality, we condition on the first hitting time of U. In the second equality, according to the PM instants, the integration interval $[0, +\infty)$ is divided into $n+2$ subintervals in order to determine the value of $\lceil (n+1)\hat{T}/T\rceil - 1$ (the number of PM actions) associated with each subinterval. As an example, if $iT/(n+1) < \tau(U) \le (i+1)T/(n+1)$, $0 \le i \le n$, then i PM actions are performed over the warranty period. The third equality follows from interchanging the expectation and integration operations and the fact that $\tau(U) = t$ implies $M(t) = U$. The fourth equality follows from Eq. (5) given that $M(0) = 0$ and $M(t) = U$.

Next, we evaluate the expected repair cost in Eq. (18). This cost can be written as Eq. (A.2) is given in Box I. In the first equality, the events $\{\sum_{j=1}^{i}\tau_j < T \le \sum_{j=1}^{i+1}\tau_j\}$, $0 \le i \le m+1$, form a partition of the sample space. The first term represents the expected repair cost when no PM effort is exerted because the usage process does not hit $U/(m+1)$ before time T. In the second term, if $\sum_{j=1}^{i}\tau_j < T \le \sum_{j=1}^{i+1}\tau_j$, then i is the number of PM actions performed before time T. To calculate the conditional expectations in the second equality, we construct a gamma bridge that is tied down at time $\sum_{h=1}^{j} t_h$ and time $\sum_{h=1}^{j+1} t_h$ for any $0 \le j \le m$. The third equality follows from tedious calculations using Eq. (5). Note that this expected repair cost has a closed-form expression involving multiple integrals. Because of the computational difficulty, we will resort to a dynamic programming model of usage-based PM.

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