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A preventive maintenance policy with usage-dependent failure rate thresholds under two-dimensional warranties

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ABSTRACT
This article considers Preventive Maintenance (PM) under a two-dimensional (2-D) warranty contract with time and usage limits. From a manufacturer's point of view, we develop a dynamic maintenance model with a random horizon to include the impact of random and dynamic usage rates on PM decisions. The model treats the cumulative amount of usage as a state variable that provides information about the failure rate and the expiration of the 2-D warranty. We characterize the optimal PM policy by a sequence of usage-dependent failure rate thresholds. Each threshold is a function of the cumulative usage. Our failure rate threshold policy chooses one of the following two actions in each period: performing perfect PM or no PM. Specifically, the manufacturer should bring the failure rate back to its original level when it exceeds the threshold in the corresponding period. This policy is also optimal under a constant usage rate. In the numerical experiments, we demonstrate the effectiveness of the proposed policy and conduct a sensitivity analysis to investigate how this policy is affected by the model parameters.

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Preventive maintenance; two-dimensional warranty; usage-dependent threshold policy; dynamic programming

1. Introduction

Sensor technology allows manufacturers to collect more data than ever before. Traditionally, manufacturers have had difficulty accessing product usage data after the sale of a product. Through the use of built-in sensors, such data from the field can be more easily and efficiently collected. Manufacturers analyze usage data to better predict future events and achieve substantial cost savings. At the same time, sensor-embedded products can provide a better experience for customers.

In the real world, many companies adopt sensor technology to develop new maintenance services. For example, in the elevator and escalator industry, KONE has recently started putting sensors in its products. In 2017, KONE launched the 24/7 Connected Services project, in which it remotely monitors product components and evaluates their health conditions in real time. This project aims to take proactive maintenance actions before a shutdown occurs (see KONE, 2019). Another example is Pivotal in the automobile industry. It looks at sensor data such as diagnostic trouble codes to predict when automobile parts will fail and to help decide when maintenance jobs should be initiated (see Ramanujam, 2016). Both companies benefit greatly from their innovative maintenance services.

This article considers Preventive Maintenance (PM) under a two-dimensional (2-D) warranty contract when manufacturers can periodically monitor the cumulative amount of product use. The interplay between PM and 2-D warranties has attracted considerable interest in the past few years. In this research area, most of the papers have focused on periodic PM (see, e.g., Wang *et al.*, 2015; Wang, Zhou and Peng, 2017; Wang, Liu, Li and Li, 2017). A maintenance interval is often measured in units of time or cumulative usage (Su and Wang 2016; Wang and Su, 2016; Wang *et al.*, 2020). Several researchers have incorporated different levels of PM effort into their policies. Shahanaghi *et al.*, (2013) determined the optimal number of PM actions and the optimal effort level of each action. Under a 2-D lease contract, Iskandar and Husniah (2017) jointly optimized the number of PM actions and the PM interval for every possible usage rate. In their model, the levels of PM effort are equal to the increments of an increasing failure rate between successive PM actions.

Limited research has been conducted with regard to the non-periodic PM in the setting of a 2-D warranty. Huang *et al.*, (2017) used different PM services including non-periodic PM to customize a 2-D extended warranty. Non-periodic PM has also been considered in some studies on designing a 2-D warranty contract (Huang and Yen, 2009; Huang *et al.*, 2013; Huang *et al.*, 2015). These studies determined the two warranty parameters under a pre-specified PM policy to maximize a manufacturer's profit.

Nearly all the above papers assumed a constant usage rate for a specific product. However, sometimes a linear usage path does not provide a good representation of a real usage process. In this case, non-constant usage rates can be taken into consideration. Several methods have been used in the literature to model non-constant usage rates, such as the logistic function (Eliashberg *et al.*, 1997), the accelerated

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Table 1. Comparison of current article with the key literature.

	Topic		Maintenance type		Usage rate		
Paper	*PM*	*2-D warranty*	*Periodic*	*Non-periodic*	*Constant*	*Dynamic*	*Setupcost?*
Wang *et al.* (2015)	✓		✓		✓		✓
Wang, Zhou and Peng (2017)	✓		✓		✓		✓
Wang, Liu, Liu and Li (2017)	✓		✓			✓	
Wang and Su (2016)	✓		✓		✓		
Su and Wang (2016)	✓		✓		✓		
Wang *et al.* (2020)	✓		✓		✓		
Shahanaghi *et al.* (2013)	✓		✓		✓		✓
Iskandar and Husniah (2017)	✓		✓		✓		✓
Huang *et al.* (2017)	✓			✓	✓		
Huang and Yen (2009)		✓		✓	✓		
Huang *et al.* (2013)		✓		✓	✓		
Huang *et al.* (2015)		✓		✓	✓		
Shi *et al.* (2019)	✓			✓			
Jack and Murthy (2002)	✓			✓			✓
This paper	✓			✓		✓	✓

failure time model (e.g., Ye *et al.*, 2013; Wang *et al.*, 2019), and a weighted prediction model (Tong *et al.*, 2017). Singpurwalla and Wilson (1993) treated usage rates as being random and dynamic. They described a general usage process by three sets of non-negative random variables, i.e., the lengths of periods of non-use and use and the usage rate in each period of use. The usage path in their setting is piecewise linear, with the slope of each piece being the realized usage rate in the corresponding period. Their method, however, appears to be analytically intractable. One way to simplify it is to exclude usage rates, as in De Jonge and Jakobsons (2018), who presented a Markov switching model where both active and idle periods are assumed to be exponentially distributed. Another way is to divide a planning horizon into periods of equal length and consider the usage rate of each period as the only source of uncertainty. This method is especially useful when manufacturers conduct periodic monitoring to understand how a product is used.

To describe the effect of usage (a time-dependent covariate) on failure time, Eliashberg *et al.*, (1997) and Singpurwalla and Wilson (1998) adopted an additive hazards model. For other types of monitoring information, many degradation models have been developed in the maintenance literature. We refer readers to Ye and Xie (2015) and Alaswad and Xiang (2017) for detailed reviews. Common continuous models include Wiener processes (e.g., Sun *et al.*, 2018; Sun *et al.*, 2020), gamma processes (e.g., Zhao *et al.*, 2019; Wu and Castro, 2020), and inverse Gaussian processes (e.g., Chen *et al.*, 2015, Shang *et al.*, 2018). Markov decision process models have been applied to maintenance problems that involve discrete degradation states (e.g., Kurt and Kharoufeh, 2010). Shi *et al.*, (2019) considered an infinite-horizon problem for the multi-level PM with complex effects. At each decision time, they determined whether to perform PM and, if so, at what level.

In this article, we consider a manufacturer who performs PM on a product or a component based on usage information obtained through periodic monitoring to reduce the cost of honoring a 2-D warranty. With this auxiliary information, the manufacturer can better assess product reliability and predict when the 2-D warranty will expire. At the beginning of each period, the cumulative usage of the product is updated to learn about its failure rate. Then the manufacturer decides whether to maintain the product in the current period and to what level to reduce the failure rate when a maintenance event occurs. We show the existence and optimality of a failure rate threshold policy. We also examine how product usage affects the decisions about PM times and levels of PM effort.

This article makes the following contributions to the literature. First, we develop a dynamic programming model that incorporates random and dynamic usage rates of a product under periodic monitoring. Second, such non-constant usage rates, together with the setup cost of PM, lead to a new PM policy with usage-dependent failure rate thresholds. Under this policy, maintenance actions may be performed non-periodically. Third, although similar decisions have been considered in some dynamic maintenance models (e.g., Jack and Murthy, 2002; Shi *et al.*, 2019), we focus on the PM in the context of a 2-D warranty. Since there is a usage limit on warranty coverage, our model is developed over a random horizon. Table 1 summarizes the current article and the key literature.

The outline of this article is as follows. In Section 2, we discuss the dynamics of a usage process and formulate a stochastic PM model. In Section 3, we show that a failure rate threshold policy is optimal. This section also describes a system under a constant usage rate. In Section 4, we give a numerical study for more insight. Finally, we conclude with a summary and some suggestions for future research.

2. Problem formulation

We consider a manufacturer who performs PM on a product or a component that is covered by a 2-D warranty contract. This warranty comes with a usage limit U and an age limit T (see Table 2 for a summary of notation). It expires when either the cumulative usage or the age of the product reaches their corresponding limits. After that, the manufacturer is not obliged to repair the product. Therefore, the 2-D warranty protects the manufacturer against high amounts of product usage, which may lead to frequent failures.

Table 2. Summary of notation.

Symbol	Definition
U	Usage limit of the 2-D warranty
T	Age limit of the 2-D warranty
R_t	Random usage rate of period t
$f(r_t)$	Density function of R_t
r	Expected value of R_t
$\underline{\lambda}$	Baseline failure rate
u_t	Cumulative usage at the beginning of period t
λ_t	Failure rate at the beginning of period t
θ_t	Failure rate after the PM of period t (decision variable)
η	Usage deterioration coefficient in the additive hazards model
k	Setup cost of PM
b	Marginal maintenance cost
c	Constant cost per repair
$L(\theta_t, u_t)$	Expected one-period repair cost
$J_t(\lambda_t, u_t)$	Manufacturer's minimum expected total cost over $\{t, t+1, \ldots, T+1\}$, given that the 2-D warranty is valid and that the PM decision in period t has not been made yet
$W_t(\theta_t, u_t)$	Manufacturer's minimum expected total cost over $\{t, t+1, \ldots, T+1\}$, given that the 2-D warranty is valid and that the PM decision in period t has just been made
$G_t(\theta_t, u_t)$	$W_t(\theta_t, u_t) - b\theta_t$
$H_t(\lambda_t, u_t)$	$\min\{G_t(\lambda_t, u_t), \min_{\underline{\lambda}} \leq \theta_t \leq \lambda_t k + G_t(\theta_t, u_t)\}$
u_t^*	Threshold of the cumulative usage in period t
t^*	Time period after which no PM actions should be performed
$s_t(u_t)$	Failure rate threshold in period t
$\gamma(u_t)$	Slope of $L(\theta_t, u_t)$
$\rho(u_t)$	Intercept of $L(\theta_t, u_t)$
$\gamma(u_t) + \alpha_t(u_t) - b$	Slope of $G_t(\theta_t, u_t)$ for any $u_t \in [u_{t+1}^*, U]$
$\beta_t(u_t)$	Intercept of $G_t(\theta_t, u_t)$ for any $u_t \in [u_{t+1}^*, U]$

2.1. State dynamics

We divide the warranty duration into T periods of equal length. We denote the usage rate in period t as R_t with a density function $f(\cdot)$ and a finite mean r. Our maintenance planning model has two state variables: the cumulative usage and the failure rate of the product. Let u_t be the cumulative usage at the beginning of period t. Then after one period, we have $u_{t+1} = u_t + R_t$. The uncertainty of R_t is resolved once the manufacturer observes the value of u_{t+1} through periodic monitoring. Given a sequence of usage rate realizations $\{r_t\}_{t=1}^T$, the cumulative usage follows a piecewise linear path.

To describe the effect of the cumulative usage on the failure rate, we assume that an additive hazards model holds (see, for example, Eliashberg *et al.*, 1997; Singpurwalla and Wilson, 1998). The failure rate at time t is given by $\lambda(t) = \lambda_0(t) + \eta M(t)$, where $\lambda_0(t)$ is a baseline failure rate that describes deterioration due to natural causes, and $M(t)$ is the cumulative usage of a product. All products or a batch of products are assumed to share the same baseline or theoretical failure rate. However, on top of the baseline failure rate, different usage levels may induce an additive failure rate that is adjusted by the cumulative usage of a particular product. We can estimate the baseline failure rate $\lambda_0(t)$ and the usage deterioration coefficient η in a laboratory test.

We next adapt the additive hazards model for use in our discrete-time setting. We assume a constant baseline failure rate $\underline{\lambda}$. That is, if the product were to be left unused, its failure rate would stay the same. A baseline failure rate that is linear in time can be easily incorporated into our failure rate function by adding a constant to the random variable R_t. Let λ_t be the failure rate at the beginning of period t. Then, the transition dynamics of the failure rate are written as $\lambda_{t+1} = \underline{\lambda} + \eta u_{t+1} = \underline{\lambda} + \eta(u_t + R_t) = \lambda_t + \eta R_t$. The last equality shows that after one period the failure rate grows by a random amount of ηR_t.

We assume that a PM action directly reduces the failure rate and that maintenance time is negligible. Let θ_t be the failure rate right after the PM of period t. This variable is referred to as the reduce-down-to level. We also assume that in each period the manufacturer can restore the failure rate to its original level; thus, we have $\underline{\lambda} \leq \theta_t \leq \lambda_t$. When $\theta_t = \underline{\lambda}$, a perfect PM action (replacement) is taken. When $\underline{\lambda} < \theta_t < \lambda_t$, PM is imperfect. When $\theta_t = \lambda_t$, PM is not performed at the beginning of this period, i.e., no technician is sent to provide PM. We describe the transition dynamics of the failure rate in the presence of PM as $\lambda_{t+1} = \lambda_t - (\lambda_t - \theta_t) + \eta R_t = \theta_t + \eta R_t$, where $\lambda_t - \theta_t$ corresponds to the maintenance effort. We assume that when reducing the failure rate by $\lambda_t - \theta_t$, the manufacturer incurs a cost of $k + b(\lambda_t - \theta_t)$, where k is the setup cost and b is the marginal maintenance cost. Jack and Murthy (2002) and Yeh and Chang (2007) made the same assumption.

2.2. Dynamic programming formulation

We now present a dynamic programming model for the PM under a 2-D warranty. At the beginning of each period, the manufacturer first observes the cumulative usage u_t to know the usage rate in the previous period. Then, the manufacturer calculates the failure rate λ_t and finds a value of the reduce-down-to level θ_t that minimizes the expected total cost over the remainder of the planning horizon. Note that one can alternatively view the maintenance effort $\lambda_t - \theta_t$ as the decision variable. However, it is more convenient to work with the reduce-down-to level.

We assume that failures follow a non-homogeneous Poisson process and that they are rectified by minimal repairs. By minimal, we mean that the failure rate stays unchanged after repairs. Each repair costs a constant c. Denote the expected repair cost of period t by $L(\theta_t, u_t)$. The first parameter of this function is θ_t—instead of λ_t—because failures in period t occur after the action of PM. From the reliability theory, we have

$$L(\theta_t, u_t) = \int_0^{U-u_t} cf(r_t) \int_0^1 (\theta_t + \eta r_t \tau) \mathrm{d}\tau \mathrm{d}r_t + \int_{U-u_t}^{+\infty} cf(r_t) \int_0^{\frac{U-u_t}{r_t}} (\theta_t + \eta r_t \tau) \mathrm{d}\tau \mathrm{d}r_t. \quad (1)$$

For each of the two terms in Equation (1), we first integrate with respect to τ, which represents the time since the beginning of period t, and then r_t, which represents a possible usage rate. The first and second inner integrals are the expected numbers of failures in period t for lines 1 and 2 in Figure 1, respectively. When the usage rate realization r_t is greater than $U - u_t$, we only count the expected number of

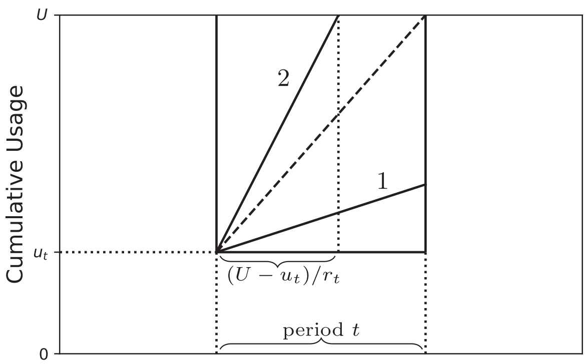

Figure 1. Two scenarios when calculating the expected one-period repair cost.

failures in the interval $[0,(U-u_t)/r_t]$ because after this period of time, the 2-D warranty expires.

Let $J_t(\lambda_t,u_t)$ denote the manufacturer's minimum expected total cost over $\{t,t+1,\ldots,T+1\}$, given that the 2-D warranty is valid and that the PM decision in period t has not yet been made. Note that period $T+1$ corresponds to the end of period T. At the decision point, we have information about λ_t and u_t. Let $W_t(\theta_t,u_t)$ denote the manufacturer's minimum expected total cost over $\{t,t+1,\ldots,T+1\}$, given that the 2-D warranty is valid and that the PM decision in period t has just been made. We now know the values of θ_t and u_t. Based on the definitions of these two cost-to-go functions, we have:

$$J_t(\lambda_t,u_t)=\min\left\{W_t(\lambda_t,u_t),\ \min_{\underline{\lambda}\le\theta_t\le\lambda_t} k+b(\lambda_t-\theta_t)+W_t(\theta_t,u_t)\right\} \tag{2}$$

and

$$W_t(\theta_t,u_t)=L(\theta_t,u_t)+\int_0^{U-u_t} J_{t+1}(\theta_t+\eta r_t,u_t+r_t)f(r_t)\mathrm{d}r_t. \tag{3}$$

Equation (2) shows the manufacturer's decision whether or not to maintain the product in period t. If no maintenance action is taken, then the first parameter of W_t will get the value of λ_t. Otherwise, the manufacturer needs to solve the inner minimization problem to reduce the failure rate to an optimal level. In this case, the manufacturer incurs a maintenance cost and the minimum expected total cost from the moment after the PM to the end of period T. The best of these two options is selected in the outer minimization.

In Equation (3), the value function $W_t(\theta_t,u_t)$ is the sum of the expected one-period repair cost and the minimum expected total cost from the next period onward. The upper limit of the integral is $U-u_t$, as $J_{t+1}(\lambda_{t+1},u_{t+1})$ exists only when the 2-D warranty is valid, i.e., when R_t is less than $U-u_t$. In essence, this equation describes a random horizon problem because the usage rate adds uncertainty to the expiration of the 2-D warranty.

For technical convenience, we define $G_t(\theta_t,u_t)$ as

$$G_t(\theta_t,u_t)=W_t(\theta_t,u_t)-b\theta_t \tag{4}$$

and $H_t(\lambda_t,u_t)$ as

$$H_t(\lambda_t,u_t)=\min\left\{G_t(\lambda_t,u_t),\ \min_{\underline{\lambda}\le\theta_t\le\lambda_t} k+G_t(\theta_t,u_t)\right\}. \tag{5}$$

Then, Equation (2) becomes

$$J_t(\lambda_t,u_t)=b\lambda_t+H_t(\lambda_t,u_t) \tag{6}$$

with the boundary conditions $J_t(\cdot\ ,U)=0$ and $J_{T+1}(\cdot\ ,\ \cdot)=0$. This reformulation is useful because we can find the optimal reduce-down-to level by comparing $G_t(\theta_t,u_t)$ at $\theta_t=\lambda_t$ with its minimum within $[\underline{\lambda},\lambda_t]$ plus k, as shown in Equation (5). One way to make such a comparison is to use the concept of k-convexity in the literature on inventory management (see, e.g., Porteus, 2002), however, it turns out that $G_t(\theta_t,u_t)$ has simpler structures than general k-convex functions.

3. Optimal maintenance policy

In this section, we first propose a failure rate threshold policy to calculate the optimal value of the reduce-down-to level θ_t. Next, we present two lemmas used in obtaining the main results of this article. We then investigate some structural properties of the value function $G_t(\theta_t,u_t)$ to prove the optimality of the proposed policy. Finally, we show similar results in a system with a constant usage rate.

Our PM policy is described by a sequence of usage-dependent failure rate thresholds. According to Equation (5), the failure rate threshold in period t for any $0\le u\le U$ is defined as

$$s_t(u)=\max\{\theta\ge\underline{\lambda}|G_t(\theta,u)\le k+G_t(\underline{\lambda},u)\}.$$

In this definition, we try to find the maximum value of the failure rate such that the above inequalities are satisfied. The threshold $s_t(u)$ depends not only on the time period, but also on the cumulative usage. As will be seen, it exhibits different properties in different subintervals of $[0,U]$.

We are now in a position to define a failure rate threshold policy as follows:

$$\theta_t=\begin{cases}\lambda_t, & \text{if } \lambda_t\le s_t(u_t),\\ \underline{\lambda}, & \text{otherwise.}\end{cases}$$

At the beginning of each period, the manufacturer evaluates the threshold $s_t(u)$ at $u=u_t$ and compares $s_t(u_t)$ with the failure rate λ_t to trigger a PM action. If $\lambda_t\le s_t(u_t)$, then maintaining the product in this period is uneconomic. On the other hand, if $\lambda_t>s_t(u_t)$, then the manufacturer should perform perfect PM to reduce the failure rate to its original level. The underlying idea of the failure rate threshold policy is to defer PM if only a low level of maintenance effort is needed so that the manufacturer will not bear the setup cost. By modifying the definition of $s_t(u)$, this policy can be generalized to the case where the lower bound of the reduce-down-to level is increasing with time.

3.1. Preliminary analysis

Before proving the optimality of the failure rate threshold policy, we need the following two lemmas to identify the structures of $G_t(\theta_t, u_t)$.

Lemma 1. *Given a fixed $u_t \in [0, U]$, the expected one-period repair cost $L(\theta_t, u_t)$ increases linearly in θ_t and can be expressed as*

$$L(\theta_t, u_t) = \gamma(u_t)\theta_t + \rho(u_t),$$

where

$$\gamma(u_t) = c\int_0^{U-u_t} f(r_t)\mathrm{d}r_t + c(U-u_t)\int_{U-u_t}^{+\infty} \frac{1}{r_t} f(r_t)\mathrm{d}r_t$$

and

$$\rho(u_t) = \frac{c\eta}{2}\int_0^{U-u_t} r_t f(r_t)\mathrm{d}r_t + \frac{c\eta(U-u_t)^2}{2}\int_{U-u_t}^{+\infty} \frac{1}{r_t} f(r_t)\mathrm{d}r_t.$$

Moreover, the slope $\gamma(u_t)$ and the intercept $\rho(u_t)$ are decreasing in u_t.

All proofs are provided in the Appendix. Given the cumulative usage, the higher the failure rate, the larger the expected one-period repair cost, due to more potential failures. As u_t gets larger, the 2-D warranty is more likely to end in the current period. Therefore, an increase in the failure rate is less severe for the manufacturer.

Lemma 2. *Consider the recursive equation for any $0 \le u \le U$:*

$$\alpha_t(u) = \int_0^{U-u} (\gamma(u + r_t) + \alpha_{t+1}(u + r_t))f(r_t)\mathrm{d}r_t \quad (7)$$

with the boundary condition $\alpha_{T+1}(u) = -\gamma(u)$. Let

$$u_t^* = \min\{0 \le u \le U | \gamma(u) + \alpha_t(u) \le b\}.$$

Then, we have:

(a) *$\gamma(u) + \alpha_t(u)$ is decreasing in u.*
(b) *$\alpha_t(u) \ge \alpha_{t+1}(u)$.*
(c) *$u_t^* \ge u_{t+1}^*$.*
(d) *If $0 \le u < u_{t+1}^*$, then $\gamma(u) + \int_{u_{t+1}^*-u}^{U-u}(\gamma(u + r_t) + \alpha_{t+1}(u + r_t))f(r_t)\mathrm{d}r_t - b\int_{u_{t+1}^*-u}^{+\infty} f(r_t)\mathrm{d}r_t \ge 0$.*

We refer to u_t^* as the threshold of the cumulative usage in period t. Its existence is guaranteed by part (a) of this lemma, and it is decreasing in t by part (c). Since u_t^* is independent of the two state variables u_t and λ_t, we can pre-compute the sequence $\{u_t^*\}_{t=1}^{T+1}$ at the beginning of the planning horizon. We use the values of u_{t+1}^* and u_t^* to partition the interval $[0, U]$ for period t into three non-overlapping subintervals, i.e., $[0, u_{t+1}^*)$, $[u_{t+1}^*, u_t^*)$, and $[u_t^*, U]$. Part (d) is proved for later use.

3.2. Properties of the value function

For any u_t that lies in the subintervals $[u_{t+1}^*, u_t^*)$ and $[u_t^*, U]$, the following theorem shows that $G_t(\theta_t, u_t)$ is a linear function of θ_t.

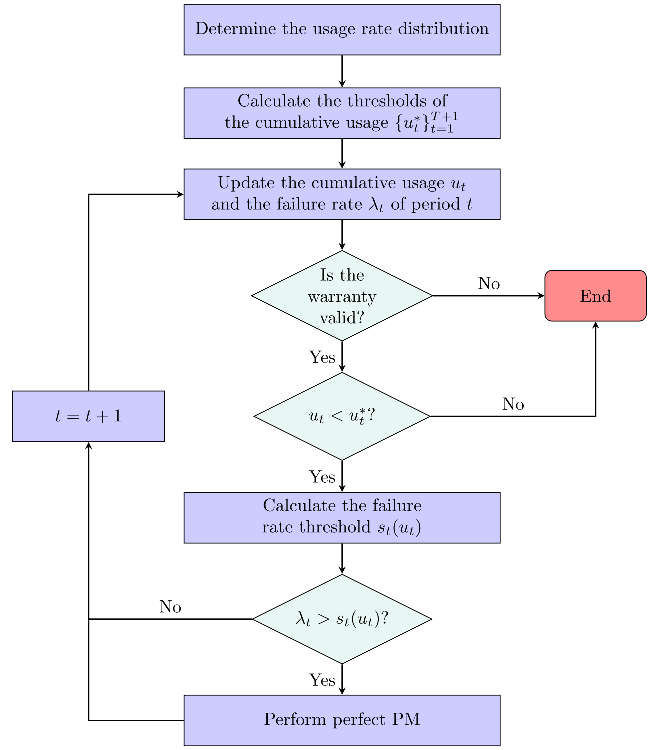

Figure 2. Implementation of the failure rate threshold policy.

Theorem 1. (Preservation of Linearity). *For $t = 1, 2, \ldots, T$ and any given $u_t \in [u_{t+1}^*, U]$, $G_t(\theta_t, u_t)$ is a linear function of the form $(\gamma(u_t) + \alpha_t(u_t) - b)\theta_t + \beta_t(u_t)$, where*

$$\begin{aligned}\beta_t(u_t) &= \rho(u_t) \\ &+ \int_0^{U-u_t} ((\gamma(u_t + r_t) + \alpha_{t+1}(u_t + r_t))\eta r_t \\ &+ \beta_{t+1}(u_t + r_t))f(r_t)\mathrm{d}r_t \end{aligned} \quad (8)$$

with the boundary condition $\beta_{T+1}(u_{T+1}) = 0$:

(a) *When $u_t^* \le u_t \le U$, $G_t(\theta_t, u_t)$ decreases linearly with θ_t. In this case, $s_t(u_t)$ is infinite, and the failure rate threshold policy is optimal in period t.*
(b) *When $u_{t+1}^* \le u_t < u_t^*$, $G_t(\theta_t, u_t)$ increases linearly with θ_t.*

With this linear expression, we can write $W_t(\theta_t, u_t) = (\gamma(u_t) + \alpha_t(u_t))\theta_t + \rho(u_t)$. Since $\gamma(u_t)$ captures the marginal increase in the expected one-period repair cost, $\alpha_t(u_t)$ represents the marginal increase in the minimum expected total cost from period $t+1$ onward for any $u_{t+1}^* \le u_t \le U$. According to Lemmas 1 and 2, these marginal changes are smaller as we move closer to the boundary of the 2-D warranty.

In part (a) of this theorem, the threshold $s_t(u_t)$ is infinite and thus can never be exceeded by the failure rate. According to the failure rate threshold policy, we always have $\theta_t = \lambda_t$. To apply this optimal policy, we need to check whether the cumulative usage meets the condition specified in part (a). Since u_t is increasing and u_t^* is decreasing in the period, it suffices for us to find when this condition is satisfied for the first time.

Theorem 2. *There exists a period $t^* = \max\{t = 1, 2, ..., T \mid u_t < u_t^*\}$ such that when $t > t^*$, it is optimal for the manufacturer not to perform PM.*

Note that this result is derived under the assumption that the manufacturer aims to minimize the expected total cost of the 2-D warranty. If the last PM activity is scheduled in a period after t^*, its cost will outweigh the reduction in the expected future cost of failures, because $b \geq \gamma(u_t) + \alpha_t(u_t)$. Therefore, after period t^*, the manufacturer decides to incur no maintenance cost, but only minimal repair costs. Under this decision rule, the product may have a high failure rate at the end of the 2-D warranty, leading to customer dissatisfaction in reality.

By definition, period t^* depends on when an increasing usage path first meets the decreasing thresholds of the cumulative usage. Hence, it is affected by the uncertainty of a usage path. The probability mass function of t^* is characterized in the following proposition.

Proposition 1. *At the beginning of period t, if u_t falls into the interval $[u_{i+1}^*, u_i^*)$, where $i \in \{t+1, t+2, ..., T\}$, then the distribution of t^* is given by*

$$\Pr(t^* = j) = \begin{cases} 1 - \Pr\left(R_t < u_{t+1}^* - u_t\right), & \text{if } j = t, \\ \Pr\left(\sum_{z=t}^{j-1} R_z < u_j^* - u_t\right) & \\ -\Pr\left(\sum_{z=t}^{j} R_z < u_{j+1}^* - u_t\right), & \text{if } j = t+1, ..., i-1, \\ \Pr\left(\sum_{z=t}^{i-1} R_z < u_i^* - u_t\right), & \text{if } j = i. \end{cases}$$

This result shows that the distribution of t^* depends on the period, the cumulative usage, and the usage rate distribution. At the beginning of each period, the manufacturer can use the observed cumulative usage to update the above probabilities. This result also shows that the probability of t^* being the current period, i.e., $\Pr(t^* = t)$, increases with time.

We now know that the failure rate threshold policy is optimal for any $u_t \in [u_t^*, U]$. When u_t is contained in the other two subintervals of $[0, U]$, we still need to show how the structural properties of $G_t(\theta_t, u_t)$ with respect to θ_t lead to the optimality of the proposed policy.

Proposition 2. *For $t = 1, 2, ..., T-1$ and any fixed vector (θ, u), $G_t(\theta, u) \geq G_{t+1}(\theta, u)$.*

The optimal cost-to-go function J_t increases as the number of periods increases, as providing the service of PM for an additional period requires an additional cost. As a result, the value function G_t is decreasing in *t*.

Theorem 3 (Preservation of Monotonicity). *The following statements are true for any given $u_t \in [0, u_t^*)$:*

(a) *$G_t(\theta_t, u_t)$ is an increasing function of θ_t, and $\lim_{\theta_t \to +\infty} G_t(\theta_t, u_t) = +\infty$.*
(b) *$s_t(u_t)$ is finite.*
(c) *The failure rate threshold policy is optimal in period t.*

When u_t is less than u_t^*, the monotonicity of $G_t(\theta_t, u_t)$ is preserved under the failure rate threshold policy. This monotonic property implies that the cost-to-go function $W_t(\theta_t, u_t)$ increases faster than $b\theta_t$. Therefore, we can view u_t^* as the minimum amount of usage below which the benefit of a one-unit reduction in the failure rate will be greater than the marginal maintenance cost. When the setup cost is zero, it pays to reduce the failure rate to its original level in all periods $t \leq t^*$; however, this trivial policy no longer applies to a maintenance problem with a non-zero setup cost. In our case, we perform perfect PM once the failure rate exceeds its threshold in the corresponding period. To look further into these usage-dependent failure rate thresholds, we need to show another property of $G_t(\theta_t, u_t)$.

Proposition 3. *For any fixed $u_t \in [0, u_t^*)$, $G_t(\theta_t, u_t)$ increases no faster than the straight line $(\gamma(u_t) + \alpha_t(u_t) - b)\theta_t + \beta_t(u_t)$.*

If the product were to be left unmaintained afterward, the value function $G_t(\theta_t, u_t)$ would have the same form as this straight line because the linearity would be preserved through dynamic programming iterations. This proposition shows the benefit of using the failure rate threshold policy over only minimally repairing the product.

Proposition 4. *The failure rate threshold $s_t(u_t)$ has the following properties:*

(a) *For $u_t \in [u_t^*, U], s_t(u_t) = +\infty$.*
(b) *For $u_t \in [u_{t+1}^*, u_t^*), s_t(u_t) = \underline{\lambda} + k/(\gamma(u_t) + \alpha_t(u_t) - b)$ and is increasing in u_t.*
(c) *For $u_t \in [0, u_{t+1}^*), s_t(u_t) \geq \underline{\lambda} + k/(\gamma(u_t) + \alpha_t(u_t) - b)$.*

Not only the cumulative usage, but also previous PM actions affect the failure rate. It is, therefore, necessary to keep track of this state variable for comparison with the threshold $s_t(u_t)$—unless $s_t(u_t)$ is infinite, i.e., when $t > t^*$. In part (b), the closed-form expression for $s_t(u_t)$ increases as the product has higher cumulative usage. Nevertheless, $s_t(u_t)$ is not monotonic in part (c), where we provide a lower bound instead. The reason is that as u_t increases to u_{t+1}^*, the value function $G_t(\theta_t, u_t)$ approaches the straight line $(\gamma(u_t) + \alpha_t(u_t) - b)\theta_t + \beta_t(u_t)$ from below, whose slope is, however, decreasing in u_t.

Our policy is similar to that of Yeh and Chang (2007), where perfect PM is performed whenever the failure rate reaches a certain threshold, but we use a sequence of usage-dependent thresholds. Figure 2 shows a flow chart of our failure rate threshold policy. Among the preparatory steps, it is critical to determine the usage rate distribution. The manufacturer can use either the population distribution or if having detailed information about a product, a more customized distribution.

3.3. Optimal policy under a constant usage rate

When random usage rates are replaced by their expected value *r*, the failure rate threshold policy is still optimal. For this deterministic system, there are two cases to consider. First, if $r \le U/T$, the 2-D warranty contract will expire at the end of period T. Second, if $r > U/T$, it will expire during period $\lceil U/r \rceil$, where $\lceil \cdot \rceil$ is the ceiling function.

In the first case, we describe the system by the following set of equations:

$$J_t(\lambda_t) = \min\left\{ W_t(\lambda_t),\ \min_{\underline{\lambda} \le \theta_t \le \lambda_t} k + b(\lambda_t - \theta_t) + W_t(\theta_t) \right\}, \quad (9)$$

$$W_t(\theta_t) = \left(\theta_t + \frac{\eta r}{2}\right)c + J_{t+1}(\theta_t + \eta r), \quad (10)$$

$$G_t(\theta_t) = W_t(\theta_t) - b\theta_t, \quad (11)$$

and

$$H_t(\lambda_t) = \min\left\{ G_t(\lambda_t),\ \min_{\underline{\lambda} \le \theta_t \le \lambda_t} k + G_t(\theta_t) \right\}. \quad (12)$$

Using Equations (11) and (12), we can write

$$J_t(\lambda_t) = b\lambda_t + H_t(\lambda_t) \quad (13)$$

with the boundary condition $J_{T+1}(\cdot) = 0$. These value functions have the same definitions as before, except that there is only one state variable. We do not record the cumulative usage because we know exactly when the 2-D warranty will end.

Define $s_t = \max\{\theta \ge \underline{\lambda} | G_t(\theta) \le k + G_t(\underline{\lambda})\}$ as the failure rate threshold of period t and let $t^* = T - \lfloor b/c \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function. After period t^*, the value function $G_t(\theta_t)$ is linearly decreasing in θ_t. In the other periods, it increases in θ_t. The manufacturer should perform perfect PM when the failure rate λ_t exceeds the threshold s_t. The proofs of these results are similar to those already presented and are omitted for brevity.

Denote a PM schedule as $(n; m_1, \ldots, m_n)$, where n is the number of perfect PM actions, and m_i is the number of periods between the $(i-1)$th and ith actions. For $n = 0$, the manufacturer incurs only repair costs. For any integer $1 \le n \le T - 1$, the manufacturer solves the integer program

$$\underset{m_i}{\text{minimize}}\ kn + b\eta r \sum_{i=1}^{n} m_i + \frac{c\eta r}{2}\sum_{i=1}^{n} m_i^2 + \frac{c\eta r}{2}\left(T - \sum_{i=1}^{n} m_i\right)^2 + c\underline{\lambda}\, T$$

$$\text{subject to } \sum_{i=1}^{n} m_i \le T - 1$$

$$m_i \ge 1 \text{ and integer}, \quad i = 1, \ldots, n.$$

The first two terms in the objective function represent the total cost of PM, and the remaining three terms correspond to the expected total cost of repairs. The optimal n is then chosen for the final PM schedule, which is not necessarily periodic because PM intervals are restricted to be integers. If we relax the integer constraints on m_i, every candidate schedule will exhibit periodicity, and thus so will the final schedule. To be specific, for a given integer $n \ge 1$, if we replace the constraints above with $\sum_{i=1}^{n} m_i \le T$ and $m_i \ge 0$ for $i = 1, \ldots, n$ and solve the resulting quadratic program, we will obtain the solution

$$m_i = \frac{T}{n+1} - \frac{b}{(n+1)c} \quad \text{if } b < Tc.$$

Otherwise, all m_i=0, in which case $n = 0$ is optimal.

In the second case, the system has a different planning horizon of length $T' + 1$, where $T' = \lceil U/r \rceil$. The terminal value function $J_{T'+1}(\lambda_{T'+1})$ is given as:

$$J_{T'+1}(\lambda_{T'+1}) = -\frac{c(rT' - U)}{r} \cdot \lambda_{T'+1} + \frac{c\eta(rT' - U)^2}{2r}.$$

The right-hand side of this equation is the negative of the extra expected repair cost incurred after the expiration of the 2-D warranty in period T'. The integer programming formulation is similar to the one in the first case.

To summarize, given a constant usage rate, PM can be optimally scheduled in either of the two ways. It is worth noting that the scheduling decision is made at the beginning of the planning horizon, which is impossible under random and dynamic usage rates because their realizations affect the implementation of the failure rate threshold policy.

4. Numerical experiments

In this section, we report the results of a base case of our PM model and study the sensitivity of the optimal policy to changes in the parameters. To compute the value functions in the model, we need to discretize the continuous random usage rate and partition the state space into a rectangular grid. Then, we can use backward induction to evaluate the value functions at the grid points and bilinear interpolation at the off-grid points.

4.1. Base case

We begin by presenting a numerical example to illustrate the analytical results discussed earlier. This example concerns the 2-D warranty of an auto component with age measured in months and usage measured in thousands of miles. For proprietary reasons, we use simulated data for illustration. The monthly usage R_t of the component is described by a truncated normal distribution on the interval [0.6,1.8] with mean $r = 1.2$ and standard deviation $\sigma = 0.4$. Moreover, we choose the following parameters: $T = 12$, $U = 12$, $\underline{\lambda} = 0$, $c = 300$, $k = 100$, $b = 1200$, and $\eta = 0.1$.

Since the thresholds of the cumulative usage can be predetermined, we first plot them as the dashed line in Figure 3. This line is horizontal from period 1 to period 8 and then drops quickly to zero. It divides the 2-D warranty into two regions: a maintenance region and a no-maintenance region (the shaded area). In the maintenance region, the manufacturer performs perfect PM when necessary. In the other region, since the failure rate thresholds are equal to infinity, PM should not be scheduled.

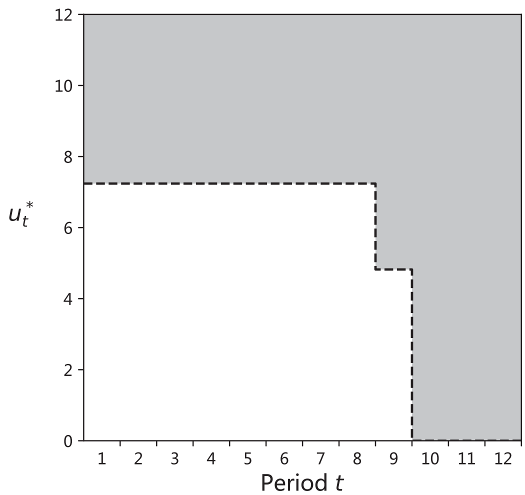

Figure 3. Values of the thresholds of the cumulative usage.

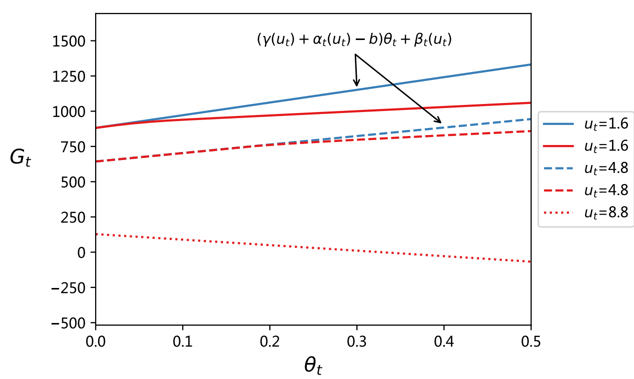

Figure 4. Value function $G_t(\theta_t, u_t)$ (red) when $t = 6$ and $u_t = 1.6,\ 4.8$, and 8.8.

Table 3. Usage rate data drawn from the truncated normal distribution.

r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}
0.89	0.64	1.17	1.40	1.13	1.18	0.63	1.19	1.67	1.54	1.41	0.80

We next turn to the value function $G_t(\theta_t, u_t)$. Figure 4 depicts how it changes in response to the reduce-down-to level θ_t for different values of u_t. The solid and dashed lines correspond to the case $u_t < u_t^*$, in which we can find the failure rate threshold by moving from the origin along the horizontal axis to the point where $G_t(\theta_t, u_t)$ increases by k. As θ_t gets larger, the gap between $G_t(\theta_t, u_t)$ and the straight line $(\gamma(u_t) + \alpha_t(u_t) - b)\theta_t + \beta_t(u_t)$ widens. However, their gap narrows as u_t gets larger, indicating a smaller benefit from future PM actions. When $u_t \geq u_t^*$, they have the same form and are both linearly decreasing in θ_t, as illustrated by the dotted line.

Figure 5 plots the failure rate threshold $s_t(u_t)$ against the cumulative usage u_t. When $u_t < u_{t+1}^*$, the threshold is always greater than or equal to $k/(\gamma(u_t) + \alpha_t(u_t) - b)$, which is consistent with part (c) of Proposition 4. These two curves coincide after u_t increases beyond a certain level because in Figure 4, $G_t(\theta_t, u_t)$ has some overlap with the straight line.

Given the usage rate data in Table 3, we use the failure rate threshold policy to determine the optimal PM schedule.

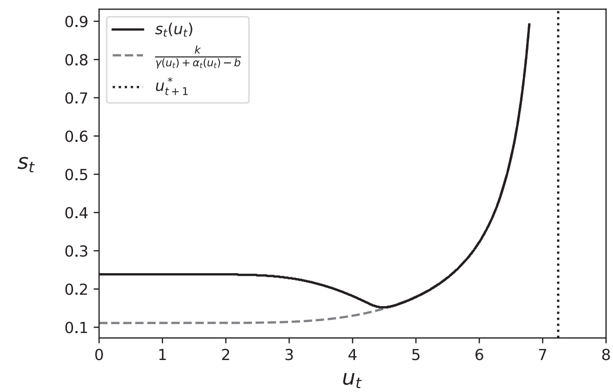

Figure 5. Failure Rate Threshold $s_t(u_t)$ when $t = 6$ and $u_{t+1}^* = 7.24$.

The cumulative usage and the failure rate are initially set to zero, and their paths are shown in Figures 6(a) and 6(b), respectively. The left figure reveals that period t^* is the eighth period. In the right figure, we see that it is more beneficial to defer PM for the first three periods. At the beginning of period 4, the failure rate is high enough to justify the setup cost so that a job of perfect PM is performed, as represented by the first dashed line. The second PM job occurs in period 6; therefore, the optimal PM schedule under our policy is not necessarily periodic. Finally, the no-maintenance region begins after period 8.

4.2. Sensitivity analysis

We next explore the sensitivity of the model to parameter variations. The following values are used: $b = 300$, 1200, 2100, 3000; $\sigma = 0.05$, 0.4, 0.9; and $k = 40$, 100, 330. We vary these parameters one at a time and set the others to their base case levels. For the last two parameters, we show the results of implementing the failure rate threshold policy over 1000 problem instances. Each instance consists of 12 usage rate realizations, which directly affect the optimal PM schedule.

We examine the impact of the cost ratio b/c on u_t^* in Table 4. An increase in the cost ratio decreases each threshold of the cumulative usage and makes more thresholds equal to zero, resulting in a larger no-maintenance region. When the marginal maintenance cost is close to the repair cost, the manufacturer has a strong incentive to carry out PM. However, for a high cost ratio, e.g., when b is 10 times larger than c, simply performing minimal repairs when there are failures is better than performing PM on the component.

Figure 7 shows a histogram of t^* for different values of the standard deviation σ. We can observe that the effect of decreasing σ is to decrease the variance of t^*. However, even if σ is relatively low, there is still some uncertainty in t^*. This is because the low variances of the independent random usage rates add up. To reduce the uncertainty, we know from Proposition 1 that updating the cumulative usage is helpful.

Another interesting quantity is the setup cost k. It is varied from 40 to 330 to show its effect on the number of PM

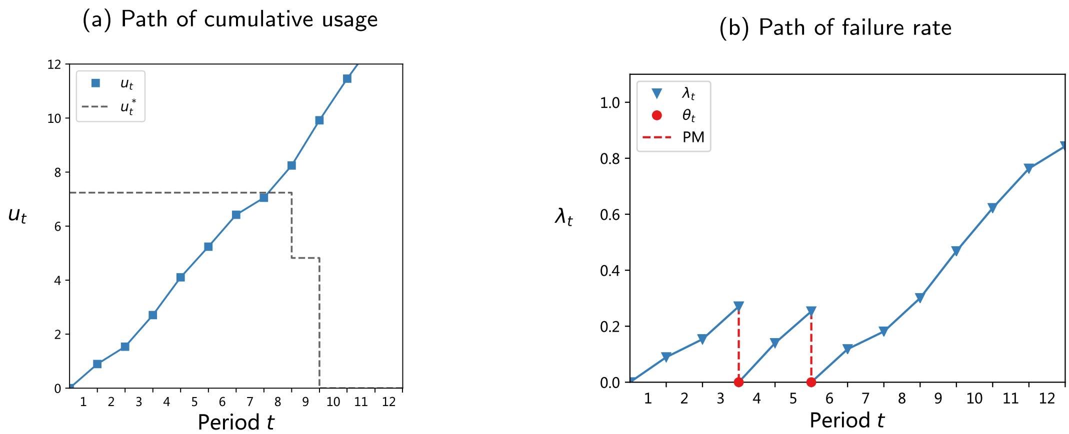

Figure 6. Schematic representation of the optimal PM schedule given the simulated data.

Table 4. Values of u_t^* for different cost ratios.

b / c	u_1^*	u_2^*	u_3^*	u_4^*	u_5^*	u_6^*	u_7^*	u_8^*	u_9^*	u_{10}^*	u_{11}^*	u_{12}^*
1	10.84	10.84	10.84	10.84	10.84	10.84	10.84	10.84	10.84	10.84	10.84	10.2
4	7.24	7.24	7.24	7.24	7.24	7.24	7.24	7.24	4.82	0	0	0
7	3.64	3.64	3.64	3.64	3.61	0	0	0	0	0	0	0
10	0.04	0	0	0	0	0	0	0	0	0	0	0

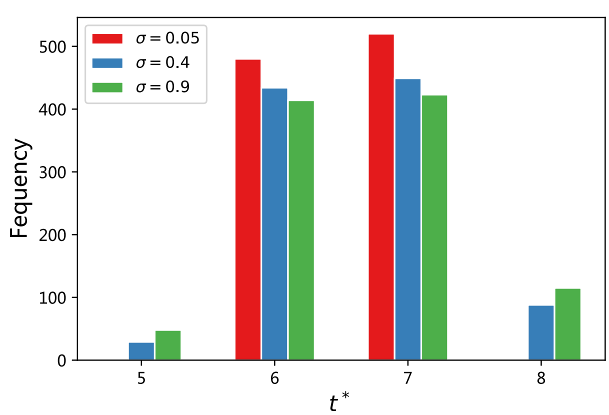

Figure 7. Histogram of t^* for 1000 usage paths when $\sigma = 0.05,\ 0.4$, and 0.9.

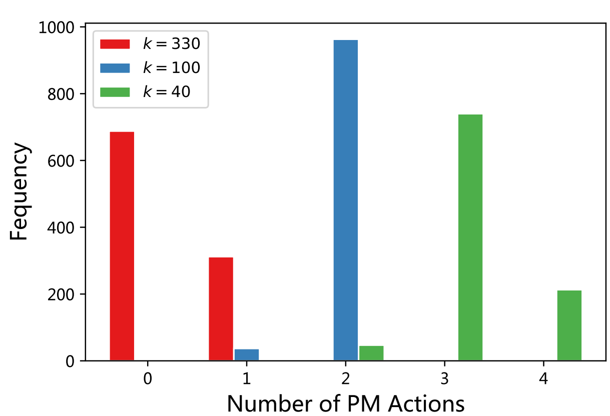

Figure 8. Histogram of the number of PM actions for 1000 usage paths when $k = 40$, 100, and 330.

actions. As seen in Figure 8, an increase in k leads to less frequent PM. Eventually, under a large setup cost, the manufacturer no longer performs PM for most usage paths because failure rate thresholds are sufficiently high.

5. Conclusions and future research

This article contributes to the burgeoning literature on the PM under a 2-D warranty by considering random and dynamic usage rates. Such usage rates lead to a failure rate threshold policy that is characterized by a sequence of usage-dependent failure rate thresholds. Depending on whether or not the failure rate is greater than its threshold in the corresponding period, this policy chooses one of the following two actions: performing perfect PM or no PM. The proposed policy is still optimal under a constant usage rate. In the numerical experiments, we show that a 2-D warranty can be divided into a maintenance region and a no-maintenance region. When the manufacturer makes decisions in the maintenance region, the setup cost plays a critical role. In the no-maintenance region, the manufacturer only performs minimal repairs.

There are several limitations to our dynamic PM model. First, we assume a linear maintenance cost for tractability. This assumption is necessary to derive Equation (6). An important extension is to consider convex or more complex cost structures. Certain structural properties of the value functions will need to be preserved under dynamic programming recursions. We conjecture that period t^* still exists, but optimal PM actions may be imperfect. Another interesting avenue for further research is to consider multi-component systems, especially when multiple failure modes are present. The state space will be expanded to capture the failure rates of all components. An opportunistic PM policy where components are jointly maintained can be developed to exploit economies of scale.

Second, we assume that the manufacturer knows the usage rate distribution at the beginning of the planning horizon. However, in many real-life situations, there is little knowledge about this distribution. Investigating the learning process of the distribution function is an appealing research direction. A Bayesian approach can be used to make inferences about the distribution parameters.

Third, we do not incorporate a customer decision-making process in the current model. Our maintenance policy serves the best interests of the manufacturer and may give customers a low utility. To alleviate this problem, the manufacturer can provide cost-sharing PM where customers pay for some of the maintenance services or partially pay for each of them. Under this type of PM, the manufacturer will reduce or even eliminate the no-maintenance region. Cost-sharing PM is commonly seen in the automobile industry and is a fruitful research direction.

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Appendix

Proof of Lemma 1. After some algebra, we have

$$\begin{aligned} L(\theta_t,u_t) &= \int_0^{U-u_t}\left(\theta_t+\frac{\eta r_t}{2}\right)cf(r_t)\mathrm{d}r_t \\ &\quad+\int_{U-u_t}^{+\infty}\frac{2\theta_t(U-u_t)+\eta(U-u_t)^2}{2r_t}\cdot \mathrm{c}f(r_t)\mathrm{d}r_t \\ &=\left(c\int_0^{U-u_t}f(r_t)\mathrm{d}r_t+c(U-u_t)\int_{U-u_t}^{+\infty}\frac{1}{r_t}f(r_t)\mathrm{d}r_t\right)\theta_t \\ &\quad+\frac{c\eta}{2}\int_0^{U-u_t}r_tf(r_t)\mathrm{d}r_t \\ &\quad+\frac{c\eta(U-u_t)^2}{2}\int_{U-u_t}^{+\infty}\frac{1}{r_t}f(r_t)\mathrm{d}r_t \\ &=\gamma(u_t)\theta_t+\rho(u_t). \end{aligned}$$

Therefore, for a fixed u_t, $L(\theta_t,u_t)$ is a linear function of θ_t. Because the slope $\gamma(u_t)\ge 0$ for any $0\le u_t\le U, L(\theta_t,u_t)$ is increasing in θ_t. By differentiating $\gamma(u_t)$ and $\rho(u_t)$ with respect to u_t, we obtain

$$\frac{\mathrm{d}\gamma(\mathrm{u_t})}{\mathrm{du_t}}=-c\int_{U-u_t}^{+\infty}\frac{1}{r_t}f(r_t)\mathrm{dr_t}\le 0$$

and

$$\frac{\mathrm{d}\rho(\mathrm{u_t})}{\mathrm{du_t}}=-c\eta(U-u_t)\int_{U-u_t}^{+\infty}\frac{1}{r_t}f(r_t)\mathrm{dr_t}\le 0.$$

□

Proof of Lemma 2. We will prove parts (a) and (b) by induction and part (c) by contradiction. From these results, part (d) follows immediately.

(a) Since $\gamma(u)+\alpha_{T+1}(u)=0$, part (a) is satisfied for period $T+1$. Suppose that $\gamma(u)+\alpha_{t+1}(u)$ is decreasing in u. Then for any $0\le u'<u\le U$, we have

$$\begin{aligned} &\gamma(u)+\alpha_t(u)-\gamma(u')-\alpha_t(u') \\ &=\int_0^{U-u}(\gamma(u+r_t)+\alpha_{t+1}(u+r_t))f(r_t)\mathrm{dr_t} \\ &\quad-\int_0^{U-u'}(\gamma(u'+r_t)+\alpha_{t+1}(u'+r_t))f(r_t)\mathrm{dr_t}+\gamma(u)-\gamma(u') \\ &=\int_0^{U-u}((\gamma(u+r_t)+\alpha_{t+1}(u+r_t))-(\gamma(u'+r_t) \\ &\quad+\alpha_{t+1}(u'+r_t)))f(r_t)\mathrm{dr_t} \\ &\quad-\int_{U-u}^{U-u'}(\gamma(u'+r_t)+\alpha_{t+1}(u'+r_t))f(r_t)\mathrm{dr_t}+\gamma(\mathrm{u})-\gamma(\mathrm{u'}) \\ &\le\int_0^{U-u}((\gamma(u+r_t)+\alpha_{t+1}(u+r_t))-(\gamma(u'+r_t) \\ &\quad+\alpha_{t+1}(u'+r_t)))f(r_t)\mathrm{dr_t} \\ &\le 0. \end{aligned}$$

The first inequality holds because $\gamma(u)-\gamma(u')\le 0$ from Lemma 1 and $\gamma(u'+r_t)+\alpha_{t+1}(u'+r_t)\ge\gamma(U)+\alpha_{t+1}(U)=0$ for any $r_t\in[U-u,U-u']$. The second inequality is due to the induction hypothesis.

(b) For $t=T$, we have $\alpha_T(u)=0\ge\alpha_{T+1}(u)=-\gamma(u)$. Suppose that $\alpha_{t+1}(u)\ge\alpha_{t+2}(u)$ for any $0\le u\le U$. Then,

$$\begin{aligned} &\alpha_t(u)-\alpha_{t+1}(u) \\ &=\int_0^{U-u}(\gamma(u+r_t)+\alpha_{t+1}(u+r_t))f(r_t)\mathrm{dr_t} \\ &\quad-\int_0^{U-u}(\gamma(u+r_{t+1})+\alpha_{t+2}(u+r_{t+1}))f(r_{t+1})\mathrm{dr_{t+1}} \\ &=\int_0^{U-u}(\alpha_{t+1}(u+r_t)-\alpha_{t+2}(u+r_t))f(r_t)\mathrm{dr_t} \\ &\ge 0. \end{aligned}$$

(c) Assume for a contradiction that there exists some t such that $0\le u_t^*<u_{t+1}^*$. Then we have

$$\gamma(u_t^*)+\alpha_t(u_t^*)\ge\gamma(u_t^*)+\alpha_{t+1}(u_t^*)>b.$$

The last inequality holds because $u_t^*<u_{t+1}^*$ by assumption and $\gamma(u)+\alpha_{t+1}(u)>b$ for any $u<u_{t+1}^*$. It, however, contradicts the fact that $\gamma(u_t^*)+\alpha_t(u_t^*)\le b$ according to the definition of u_t^*.

(d) Notice that

$$\begin{aligned} &\gamma(u)+\int_{u_{t+1}^*-u}^{U-u}(\gamma(u+r_t)+\alpha_{t+1}(u+r_t))f(r_t)\mathrm{dr_t} \\ &\quad-\mathrm{b}\int_{\mathrm{u_{t+1}^*-u}}^{+\infty}\mathrm{f(r_t)dr_t} \\ &=\gamma(u)+\alpha_t(u)-\int_0^{u_{t+1}^*-u}(\gamma(u+r_t)+\alpha_{t+1}(u+r_t))f(r_t)\mathrm{dr_t} \\ &\quad-\mathrm{b}\int_{\mathrm{u_{t+1}^*-u}}^{+\infty}\mathrm{f(r_t)dr_t} \\ &=\left(\int_0^{u_{t+1}^*-u}f(r_t)\mathrm{dr_t}+\int_{\mathrm{u_{t+1}^*-u}}^{+\infty}\mathrm{f(r_t)dr_t}\right)(\gamma(u)+\alpha_t(u)) \\ &\quad-\int_0^{u_{t+1}^*-u}(\gamma(u+r_t)+\alpha_{t+1}(u+r_t))f(r_t)\mathrm{dr_t} \\ &\quad-\mathrm{b}\int_{\mathrm{u_{t+1}^*-u}}^{+\infty}\mathrm{f(r_t)dr_t} \\ &=\int_0^{u_{t+1}^*-u}((\gamma(u)+\alpha_t(u))-(\gamma(u+r_t)+\alpha_{t+1}(u+r_t)))f(r_t)\mathrm{dr_t} \\ &\quad+(\gamma(u)+\alpha_t(u)-b)\int_{u_{t+1}^*-u}^{+\infty}f(r_t)\mathrm{dr_t} \\ &\ge 0. \end{aligned}$$

The first equality follows from Equation (7). The inequality holds because $\gamma(u)+\alpha_t(u)\ge\gamma(u+r_t)+\alpha_t(u+r_t)\ge\gamma(u+r_t)+\alpha_{t+1}(u+r_t)$ and $\gamma(u)+\alpha_t(u)-b>0$ for any $0\le u<u_{t+1}^*\le u_t^*$. □

Proof of Theorem 1. For $t=T$, $G_T(\theta_T,u_T)=(\gamma(u_T)-b)\theta_T+\rho(u_T)$ when $0=u_{T+1}^*\le u_T\le U$. (Note that $\alpha_T(u_T)=0$ and $\beta_T(u_T)=\rho(u_T)$.) Assume by induction that $G_{t+1}(\theta_{t+1},u_{t+1})=(\gamma(u_{t+1})+\alpha_{t+1}(u_{t+1})-b)\theta_{t+1}+\beta_{t+1}(u_{t+1})$ for any $u_{t+1}\in[u_{t+2}^*,U]$. When $u_{t+1}\in[u_{t+1}^*,U]$, the slope of $G_{t+1}(\theta_{t+1},u_{t+1})$ is non-positive by the definition of u_{t+1}^*, and thus $s_{t+1}(u_{t+1})=+\infty$. For any λ_{t+1}, we have

$$\begin{aligned} &H_{t+1}(\lambda_{t+1},u_{t+1}) \\ &=\min\{G_{t+1}(\lambda_{t+1},u_{t+1}),\min_{\underline{\lambda}}\le\theta_{t+1}\le\lambda_{t+1}k+G_{t+1}(\theta_{t+1},u_{t+1})\} \\ &=G_{t+1}(\lambda_{t+1},u_{t+1}). \end{aligned}$$

The last equality holds because $G_{t+1}(\theta_{t+1},u_{t+1})$ decreases linearly with θ_{t+1}. This equality shows that $\theta_{t+1}=\lambda_{t+1}$ for any $\lambda_{t+1}\le s_{t+1}(u_{t+1})$; hence, the failure rate threshold policy is optimal in period $t+1$ when $u_{t+1}\in[u_{t+1}^*,U]$. For period t and any $u_t\in[u_{t+1}^*,U]$,

$$
\begin{aligned}
&G_t(\theta_t, u_t)\\
&= -b\theta_t + L(\theta_t, u_t) + \int_0^{U-u_t} (b(\theta_t + \eta r_t) + H_{t+1}(\theta_t + \eta r_t, u_t + r_t))f(r_t)\mathrm{dr_t}\\
&= -b\theta_t + L(\theta_t, u_t) + \int_0^{U-u_t} (b(\theta_t + \eta r_t) + G_{t+1}(\theta_t + \eta r_t, u_t + r_t))f(r_t)\mathrm{dr_t}\\
&= -b\theta_t + L(\theta_t, u_t) + \int_0^{U-u_t} ((\gamma(u_t + r_t) + \alpha_{t+1}(u_t + r_t))(\theta_t + \eta r_t)\\
&\quad +\beta_{t+1}(u_t + r_t))f(r_t)\mathrm{dr_t}\\
&= \Big(\gamma(u_t) + \textstyle\int_0^{U-u_t} (\gamma(u_t + r_t) + \alpha_{t+1}(u_t + r_t))f(r_t)\mathrm{dr_t} - \mathrm{b}\Big)\theta_t + \rho(u_t)\\
&\quad + \int_0^{U-u_t} ((\gamma(u_t + r_t) + \alpha_{t+1}(u_t + r_t))\eta r_t + \beta_{t+1}(u_t + r_t))f(r_t)\mathrm{dr_t}\\
&= (\gamma(u_t) + \alpha_t(u_t) - b)\theta_t + \beta_t(u_t).
\end{aligned}
$$

The second equality holds because $u^*_{t+1} \le u_t \le u_t + r_t \le U$ for any $r_t \in [0, U - u_t]$. After substituting the linear expression of $G_{t+1}(\theta_t + \eta r_t, u_t + r_t)$, we arrive at the third equality. The last equality shows that the linear structure is preserved under the failure rate threshold policy.

To determine the sign of the slope of $G_t(\theta_t, u_t)$, we further divide the interval $[u^*_{t+1}, U]$ into two subintervals $[u^*_{t+1}, u^*_t)$ and $[u^*_t, U]$. When $u^*_t \le u_t \le U$, the definition of u^*_t implies that $G_t(\theta_t, u_t)$ is linearly decreasing in θ_t. Therefore, the failure rate threshold policy is optimal in period t. When $u^*_{t+1} \le u_t < u^*_t, G_t(\theta_t, u_t)$ is linearly increasing in θ_t. □

Proof of Theorem 2. Take $t^* = 0$ if the set in the definition of t^* is empty, i.e., when $u^*_1 = 0$. For any $t > t^*$, we have $u^*_t \le u^*_{t+1} \le u_{t^*+1} \le u_t$. The first inequality follows from part (c) of Lemma 2. The second inequality is due to the definition of t^*. The third inequality holds because u_t is increasing in t. Since the condition in part (a) of Theorem 1 is satisfied for $t = t^* + 1, t^* + 2, \ldots, T$, we know that the manufacturer chooses not to perform PM in these periods. □

Proof of Proposition 1. If u_t falls into the interval $[u^*_{i+1}, u^*_i)$, then t^* takes values in the set $\{t, t+1, \ldots, i\}$. To describe its distribution, we first prove by induction that $\Pr(t^* \ge j) = \Pr(u_t + \sum_{z=t}^{j-1} R_z < u^*_j)$ for $j = t+1, t+2, \ldots, i$. When $j = i$, we have $\Pr(t^* \ge i) = \Pr(t^* = i) = \Pr(u_t + \sum_{z=t}^{i-1} R_z < u^*_i)$. Suppose that $\Pr(t^* \ge j+1) = \Pr(u_t + \sum_{z=t}^{j} R_z < u^*_{j+1})$. Using the law of total probability, this equation can be rewritten as

$$
\begin{aligned}
&\Pr(t^* \ge j+1)\\
&= \Pr\left(R_j < u^*_{j+1} - u_t - \sum_{z=t}^{j-1} R_z\right)\\
&= \Pr\left(u_t + \sum_{z=t}^{j-1} R_z < u^*_{j+1}\right)\Pr\left(R_j < u^*_{j+1} - u_t - \sum_{z=t}^{j-1} R_z \Big| u_t + \sum_{z=t}^{j-1} R_z < u^*_{j+1}\right)\\
&\quad + \Pr\left(u_t + \sum_{z=t}^{j-1} R_z \ge u^*_{j+1}\right)\Pr\left(R_j < u^*_{j+1} - u_t - \sum_{z=t}^{j-1} R_z \Big| u_t + \sum_{z=t}^{j-1} R_z \ge u^*_{j+1}\right)\\
&= \Pr\left(u_t + \sum_{z=t}^{j-1} R_z < u^*_{j+1}\right)\Pr\left(R_j < u^*_{j+1} - u_t - \sum_{z=t}^{j-1} R_z \Big| u_t + \sum_{z=t}^{j-1} R_z < u^*_{j+1}\right),
\end{aligned}
$$

where the last conditional probability in the second equality is zero. Then we have

$$
\begin{aligned}
\Pr(t^* \ge j) &= \Pr(t^* \ge j+1) + \Pr(t^* = j)\\
&= \Pr(t^* \ge j+1) + \Pr\left(u^*_{j+1} \le u_t + \sum_{z=t}^{j-1} R_z < u^*_j\right)\\
&\quad + \Pr\left(u_t + \sum_{z=t}^{j-1} R_z < u^*_{j+1}\right)\Pr\left(R_j \ge u^*_{j+1} - u_t - \sum_{z=t}^{j-1} R_z \Big| u_t + \sum_{z=t}^{j-1} R_z < u^*_{j+1}\right)\\
&= \Pr\left(u_t + \sum_{z=t}^{j-1} R_z < u^*_{j+1}\right) + \Pr\left(u^*_{j+1} \le u_t + \sum_{z=t}^{j-1} R_z < u^*_j\right)\\
&= \Pr\left(u_t + \sum_{z=t}^{j-1} R_z < u^*_j\right),
\end{aligned}
$$

where the second equality holds because $\Pr(t^* = j)$ is the sum of the probability that $u^*_{j+1} \le u_j < u^*_j$ and the probability that $u_j < u^*_{j+1}$ and $u_{j+1} \ge u^*_{j+1}$. To simplify $\Pr(t^* = j)$, we proceed as follows:

$$
\begin{aligned}
\Pr(t^* = j) &= \Pr(t^* \ge j) - \Pr(t^* \ge j+1)\\
&= \Pr\left(u_t + \sum_{z=t}^{j-1} R_z < u^*_j\right) - \Pr\left(u_t + \sum_{z=t}^{j} R_z < u^*_{j+1}\right)\\
&= \Pr\left(\sum_{z=t}^{j-1} R_z < u^*_j - u_t\right) - \Pr\left(\sum_{z=t}^{j} R_z < u^*_{j+1} - u_t\right).
\end{aligned}
$$

We conclude the proof by checking that

$$
\Pr(t^* = t) = \Pr\big(R_t \ge u^*_{t+1} - u_t\big) = 1 - \Pr\big(R_t < u^*_{t+1} - u_t\big).
$$

Proof of Proposition 2. We first show inductively that $J_t(\lambda, u) \ge J_{t+1}(\lambda, u)$ for any fixed vector (λ, u). When $t = T$, $J_T(\lambda, u) - J_{T+1}(\lambda, u) = J_T(\lambda, u) \ge 0$. Assume by induction that $J_{t+1}(\lambda, u) \ge J_{t+2}(\lambda, u)$ for period $t+1$. Since

$$
G_t(\theta, u) = L(\theta, u) + \int_0^{U-u} J_{t+1}(\theta + \eta r_t, u + r_t)f(r_t)\mathrm{dr_t} - \mathrm{b}\theta,
$$

we have

$$
G_t(\theta, u) - G_{t+1}(\theta, u) = \int_0^{U-u} (J_{t+1}(\theta + \eta r_t, u + r_t) - J_{t+2}(\theta + \eta r_t, u + r_t))f(r_t)\mathrm{dr_t} \ge 0.
$$

For period t,

$$
\begin{aligned}
&J_t(\lambda, u) - J_{t+1}(\lambda, u)\\
&= H_t(\lambda, u) - H_{t+1}(\lambda, u)\\
&= \min\Big\{G_t(\lambda, u),\ \min\nolimits_{\underline{\lambda} \le \theta \le \lambda} k + G_t(\theta, u)\Big\}\\
&\quad -\min\Big\{G_{t+1}(\lambda, u),\ \min\nolimits_{\underline{\lambda} \le \theta \le \lambda} k + G_{t+1}(\theta, u)\Big\}\\
&\ge \min\Big\{G_{t+1}(\lambda, u),\ \min\nolimits_{\underline{\lambda} \le \theta \le \lambda} k + G_{t+1}(\theta, u)\Big\}\\
&\quad -\min\Big\{G_{t+1}(\lambda, u),\ \min\nolimits_{\underline{\lambda} \le \theta \le \lambda} k + G_{t+1}(\theta, u)\Big\} = 0.
\end{aligned}
$$

The inequality holds because $G_t(\lambda, u) \ge G_{t+1}(\lambda, u)$ and

$$
\begin{aligned}
\min\nolimits_{\underline{\lambda}} \le \theta \le \lambda k + G_t(\theta, u) &= k + G_t(\theta^*_t, u) \ge k + G_{t+1}(\theta^*_t, u)\\
&\ge \min\nolimits_{\underline{\lambda}} \le \theta \le \lambda k + G_{t+1}(\theta, u),
\end{aligned}
$$

where $\theta^*_t = \arg\min_{\underline{\lambda}} \le \theta \le \lambda k + G_t(\theta, u)$. Since $J_t(\lambda, u) \ge J_{t+1}(\lambda, u)$, it follows that $G_{t-1}(\theta, u) \ge G_t(\theta, u)$. □

Proof of Theorem 3. The proof is based on an induction argument built around part (a) of this theorem. For period T, $G_T(\theta_T, u_T) = (\gamma(u_T) - b)\theta_T + \rho(u_T)$ is strictly increasing in θ_T for any $0 \le u_T < u^*_T$, and hence $\lim_{\theta_T \to +\infty} G_T(\theta_T, u_T) = +\infty$. Suppose that part (a) is true for period $t+1$. Part (b) follows immediately from part (a). To obtain the optimality of the failure rate threshold policy, we examine $H_{t+1}(\lambda_{t+1}, u_{t+1})$ in the following two cases: (i) for $\lambda_{t+1} > s_{t+1}(u_{t+1})$, we have

$$
\begin{aligned}
H_{t+1}(\lambda_{t+1}, u_{t+1}) &= \min\Big\{G_{t+1}(\lambda_{t+1}, u_{t+1}),\ \min\nolimits_{\underline{\lambda} \le \theta_{t+1} \le \lambda_{t+1}} k + G_{t+1}(\theta_{t+1}, u_{t+1})\Big\}\\
&= \min\{G_{t+1}(\lambda_{t+1}, u_{t+1}), k + G_{t+1}(\underline{\lambda}, u_{t+1})\}\\
&= k + G_{t+1}(\underline{\lambda}, u_{t+1}).
\end{aligned}
$$

The second equality follows from the induction hypothesis that $G_{t+1}(\theta_{t+1}, u_{t+1})$ is increasing in θ_{t+1}. The last equality is due to the definition of $s_{t+1}(u_{t+1})$. (ii) For $\lambda_{t+1} \le s_{t+1}(u_{t+1})$, similarly, we have $H_{t+1}(\lambda_{t+1}, u_{t+1}) = G_{t+1}(\lambda_{t+1}, u_{t+1})$.

We next establish that $H_{t+1}(\lambda_{t+1}, u_{t+1})$ is increasing in λ_{t+1} when $0 \le u_{t+1} < u^*_{t+1}$. Suppose that $\lambda'_{t+1} < \lambda_{t+1}$. There are three cases to consider:

(i) If $s_{t+1}(u_{t+1}) < \lambda'_{t+1} < \lambda_{t+1}$, then $H_{t+1}(\lambda_{t+1}, u_{t+1}) - H_{t+1}(\lambda'_{t+1}, u_{t+1}) = k + G_{t+1}(\underline{\lambda}, u_{t+1}) - k - G_{t+1}(\underline{\lambda}, u_{t+1}) = 0$.

(ii) If $\lambda'_{t+1} \le s_{t+1}(u_{t+1}) < \lambda_{t+1}$, then $H_{t+1}(\lambda_{t+1}, u_{t+1}) - H_{t+1}(\lambda'_{t+1}, u_{t+1}) = k + G_{t+1}(\underline{\lambda}, u_{t+1}) - G_{t+1}(\lambda'_{t+1}, u_{t+1}) \ge 0$ by the definition of $s_{t+1}(u_{t+1})$.

(iii) If $\lambda'_{t+1} < \lambda_{t+1} \le s_{t+1}(u_{t+1})$, then $H_{t+1}(\lambda_{t+1}, u_{t+1}) - H_{t+1}(\lambda'_{t+1}, u_{t+1}) = G_{t+1}(\lambda_{t+1}, u_{t+1}) - G_{t+1}(\lambda'_{t+1}, u_{t+1}) \ge 0$ from the induction hypothesis.

After knowing the monotonicity of $H_{t+1}(\lambda_{t+1}, u_{t+1})$, we can show that $G_t(\theta_t, u_t)$ is increasing with respect to θ_t in two cases. Consider first the case $0 \le u_t < u^*_{t+1}$. For any $\theta'_t < \theta_t$, we have

$$\begin{aligned}
&G_t(\theta_t, u_t) - G_t(\theta'_t, u_t) \\
&= (\gamma(u_t) - b)(\theta_t - \theta'_t) + \int_0^{U-u_t} (J_{t+1}(\theta_t + \eta r_t, u_t + r_t) \\
&-J_{t+1}(\theta'_t + \eta r_t, u_t + r_t))f(r_t)\mathrm{dr_t} \\
&= (\gamma(u_t) - b)(\theta_t - \theta'_t) + \int_{u^*_{t+1}-u_t}^{U-u_t} (J_{t+1}(\theta_t + \eta r_t, u_t + r_t) \\
&-J_{t+1}(\theta'_t + \eta r_t, u_t + r_t))f(r_t)\mathrm{dr_t} \\
&\quad + \int_0^{u^*_{t+1}-u_t} (J_{t+1}(\theta_t + \eta r_t, u_t + r_t) - J_{t+1}(\theta'_t + \eta r_t, u_t + r_t))f(r_t)\mathrm{dr_t} \\
&= (\gamma(u_t) - b)(\theta_t - \theta'_t) + \int_{u^*_{t+1}-u_t}^{U-u_t} (\gamma(u_t + r_t) + \alpha_{t+1}(u_t + r_t))(\theta_t - \theta'_t)f(r_t)\mathrm{dr_t} \\
&\quad + \int_0^{u^*_{t+1}-u_t} (b(\theta_t - \theta'_t) + H_{t+1}(\theta_t + \eta r_t, u_t + r_t) - H_{t+1}(\theta'_t + \eta r_t, u_t + r_t))f(r_t)\mathrm{dr_t} \\
&= \left(\gamma(u_t) + \int_{u^*_{t+1}-u_t}^{U-u_t} (\gamma(u_t + r_t) + \alpha_{t+1}(u_t + r_t))f(r_t)\mathrm{dr_t} - \mathrm{b}\int_{\mathrm{u}^*_{t+1}-\mathrm{u_t}}^{+\infty} \mathrm{f(r_t)dr_t}\right)(\theta_t - \theta'_t) \\
&\quad + \int_0^{u^*_{t+1}-u_t} (H_{t+1}(\theta_t + \eta r_t, u_t + r_t) - H_{t+1}(\theta'_t + \eta r_t, u_t + r_t))f(r_t)\mathrm{dr_t} \ge 0.
\end{aligned}$$

The third equality holds because

$$\begin{aligned}
J_{t+1}(\lambda_{t+1}, u_{t+1}) &= b\lambda_{t+1} + H_{t+1}(\lambda_{t+1}, u_{t+1}) \\
&= b\lambda_{t+1} + G_{t+1}(\lambda_{t+1}, u_{t+1}) \\
&= (\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}))\lambda_{t+1} + \beta_{t+1}(u_{t+1})
\end{aligned}$$

for any $u^*_{t+1} \le u_{t+1} \le U$. The inequality is due to part (d) of Lemma 2 and the fact that $H_{t+1}(\lambda_{t+1}, u_{t+1})$ is increasing in λ_{t+1} when $0 \le u_{t+1} < u^*_{t+1}$. Given any $0 \le u_t < u^*_{t+1}$, we have $\lim_{\theta_t \to +\infty} G_t(\theta_t, u_t) \ge \lim_{\theta_t \to +\infty} G_{t+1}(\theta_t, u_t) = +\infty$ by Proposition 2. Now consider the case $u^*_{t+1} \le u_t < u^*_t$. Part (a) follows immediately from Theorem 1. This concludes the induction argument for the first part of this theorem. The proofs of parts (b) and (c) are omitted because they are similar to those for period $t+1$. □

Proof of Proposition 3. Equivalently, we show that for any $0 \le u_t < u^*_t$ and $\theta'_t < \theta_t$,

$$\begin{aligned}
&G_t(\theta_t, u_t) - G_t(\theta'_t, u_t) - ((\gamma(u_t) + \alpha_t(u_t) - b)\theta_t + \beta_t(u_t) \\
&-(\gamma(u_t) + \alpha_t(u_t) - b)\theta'_t - \beta_t(u_t)) \\
&= G_t(\theta_t, u_t) - G_t(\theta'_t, u_t) - (\gamma(u_t) + \alpha_t(u_t) - b)(\theta_t - \theta'_t) \\
&\le 0.
\end{aligned}$$

In period T, $G_T(\theta_T, u_T) - G_T(\theta'_T, u_T) - (\gamma(u_T) + \alpha_T(u_T) - b)(\theta_T - \theta'_T) = 0$ when $0 = u^*_{T+1} \le u_T < u^*_T$ and $\theta'_T < \theta_T$. Suppose that this is true for period $t+1$. We now demonstrate that

$$\begin{aligned}
&H_{t+1}(\lambda_{t+1}, u_{t+1}) - H_{t+1}(\lambda'_{t+1}, u_{t+1}) \\
&\quad - (\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b)(\lambda_{t+1} - \lambda'_{t+1}) \\
&\le 0
\end{aligned}$$

for any $0 \le u_{t+1} < u^*_{t+1}$ and $\lambda'_{t+1} < \lambda_{t+1}$ by investigating the following three cases:

(i) If $s_{t+1}(u_{t+1}) < \lambda'_{t+1} < \lambda_{t+1}$, then

$$\begin{aligned}
&H_{t+1}(\lambda_{t+1}, u_{t+1}) - H_{t+1}(\lambda'_{t+1}, u_{t+1}) \\
&\quad - (\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b)(\lambda_{t+1} - \lambda'_{t+1}) \\
&\quad = k + G_{t+1}(\underline{\lambda}, u_{t+1}) - k - G_{t+1}(\underline{\lambda}, u_{t+1}) \\
&\qquad - (\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b)(\lambda_{t+1} - \lambda'_{t+1}) \\
&\quad = -(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b)(\lambda_{t+1} - \lambda'_{t+1}) \le 0
\end{aligned}$$

by the definition of u^*_{t+1}.

(ii) If $\lambda'_{t+1} \le s_{t+1}(u_{t+1}) < \lambda_{t+1}$, then

$$\begin{aligned}
&H_{t+1}(\lambda_{t+1}, u_{t+1}) - H_{t+1}(\lambda'_{t+1}, u_{t+1}) \\
&\quad -(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b)(\lambda_{t+1} - \lambda'_{t+1}) \\
&= k + G_{t+1}(\underline{\lambda}, u_{t+1}) - G_{t+1}(\lambda'_{t+1}, u_{t+1}) \\
&\quad -(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b)(\lambda_{t+1} - \lambda'_{t+1}) \\
&\le G_{t+1}(s_{t+1}(u_{t+1}), u_{t+1}) - G_{t+1}(\lambda'_{t+1}, u_{t+1}) \\
&\quad -(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b)\left(s_{t+1}(u_{t+1}) - \lambda'_{t+1}\right) \\
&\le 0.
\end{aligned}$$

The first inequality holds because $k + G_{t+1}(\underline{\lambda}, u_{t+1}) = G_{t+1}(s_{t+1}(u_{t+1}), u_{t+1})$ and $\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b > 0$ when $0 \le u_{t+1} < u^*_{t+1}$. The second inequality follows from the induction hypothesis.

(iii) If $\lambda'_{t+1} < \lambda_{t+1} \le s_{t+1}(u_{t+1})$, then

$$\begin{aligned}
&H_{t+1}(\lambda_{t+1}, u_{t+1}) - H_{t+1}(\lambda'_{t+1}, u_{t+1}) \\
&\quad -(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b)(\lambda_{t+1} - \lambda'_{t+1}) \\
&\quad = G_{t+1}(\lambda_{t+1}, u_{t+1}) - G_{t+1}(\lambda'_{t+1}, u_{t+1}) \\
&\qquad - (\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b)(\lambda_{t+1} - \lambda'_{t+1}) \\
&\quad \le 0
\end{aligned}$$

from the induction hypothesis.

For period t, we consider the cases $0 \le u_t < u^*_{t+1}$ and $u^*_{t+1} \le u_t < u^*_t$. In the first case, for any $\theta'_t < \theta_t$, we have

$$\begin{aligned}
&G_t(\theta_t, u_t) - G_t(\theta'_t, u_t) - (\gamma(u_t) + \alpha_t(u_t) - b)(\theta_t - \theta'_t) \\
&= \left(\gamma(u_t) + \int_{u^*_{t+1}-u_t}^{U-u_t} (\gamma(u_t + r_t) + \alpha_{t+1}(u_t + r_t))f(r_t)\mathrm{dr_t} - \mathrm{b}\int_{\mathrm{u}^*_{t+1}-\mathrm{u_t}}^{+\infty} \mathrm{f(r_t)dr_t}\right)(\theta_t - \theta'_t) \\
&\quad + \int_0^{u^*_{t+1}-u_t} (H_{t+1}(\theta_t + \eta r_t, u_t + r_t) - H_{t+1}(\theta'_t + \eta r_t, u_t + r_t))f(r_t)\mathrm{dr_t} \\
&\quad - (\gamma(u_t) + \alpha_t(u_t) - b)(\theta_t - \theta'_t) \\
&= \int_0^{u^*_{t+1}-u_t} (H_{t+1}(\theta_t + \eta r_t, u_t + r_t) - H_{t+1}(\theta'_t + \eta r_t, u_t + r_t) \\
&\quad - (\gamma(u_t + r_t) + \alpha_{t+1}(u_t + r_t) - b)(\theta_t - \theta'_t))f(r_t)\mathrm{dr_t} \\
&\le 0.
\end{aligned}$$

The first equality can be seen from the proof of Theorem 3, and the second equality follows from Equation (7). The inequality is substantiated by the statement about H_{t+1}. The proof for the second case is trivial and thus omitted. □

Proof of Proposition 4. Part (a) follows from Theorem 1. As for part (b), since $G_t(\theta_t, u_t)$ is linear and strictly increasing in $\theta_t$when $u_t \in [u^*_{t+1}, u^*_t)$, if we start at $\theta_t = \underline{\lambda}$ and take a step of $k/(\gamma(u_t) + \alpha_t(u_t) - b)$, then we will arrive at $\theta_t = s_t(u_t)$. Differentiating $s_t(u_t)$ with respect to $u_t$yields

$$\frac{\mathrm{ds_t(u_t)}}{\mathrm{du_t}} = \frac{-k}{(\gamma(u_t) + \alpha_t(u_t) - b)^2} \cdot \frac{\mathrm{d}\ (\gamma(\mathrm{u_t}) + \alpha_t(\mathrm{u_t}))}{\mathrm{du_t}} \ge 0,$$

where the inequality is due to part (a) of Lemma 2. Next, we establish the third part of this proposition. By Proposition 3, we have $G_t(\theta_t, u_t) - G_t(\underline{\lambda}, u_t) - (\gamma(u_t) + \alpha_t(u_t) - b)(\theta_t - \underline{\lambda}) \le 0$ for any $0 \le u_t < u^*_{t+1}$. Letting $\theta_t = s_t(u_t)$ yields $G_t(s_t(u_t), u_t) - G_t(\underline{\lambda}, u_t) = k \le (\gamma(u_t) + \alpha_t(u_t) - b)(s_t(u_t) - \underline{\lambda})$, or equivalently, $s_t(u_t) \ge \underline{\lambda} + k/(\gamma(u_t) + \alpha_t(u_t) - b)$. □